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Optimality and Duality in Generalized Pareto Minimization in Complete Normed Spaces

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Abstract

This paper deals with the minimization of a generalized Pareto problem defined on complete normed spaces. Some sufficient optimality conditions are given. Some results on weak and strong duality are derived.

Key words: Complete normed spaces, Duality, Generalized Pareto minimization, Optimality conditions, Type I functions

1. Introduction

Let X, Y and Z be three complete normed linear spaces, U and V are subsets of X and Z respectively.

Consider the following mathematical programming problem:

$$minimize\ \{g(u): u \in U, -h(u) \in V\},$$

where $g: X \to Y$ and $h: X \to Z$.

Minami [1] considered an ordinary multi-objective program on a Banach space, in which objective functions and constraint functions were locally Lipschitzian but not always convex, and derived Kuhn-Tucker forms given by Clarke's generalized gradients [2] as necessary conditions for weak Pareto optimum.

The above mathematical programming problem was studied by Clarke [3], Minami [4], Craven [5], Reiland [6], Lee [7], Bhatia et al. [8], Giorgi et al. [9], Mishra et al. [10], Liu [11], Mishra [12] and Kim [13], when Y and Z are finite-dimensional normed linear spaces, g and h are locally Lipschitz functions.

Abdouni et al. [14] and Coladas et al. [15] were considered the Lipschitz infinite dimensional cases.

Brandao et al. [16] studied multi-objective mathematical programming problems with non-differentiable strongly compact Lipschitz functions defined on general complete normed linear spaces.

Brandao et al. [16] established KKT type conditions and Mond-Weir type duality results under a Slater-type condition and an invexity notion for mappings defined between complete normed linear spaces.

In this paper, we extend the concept of type I functions [17], pseudo-Type I and quasi-Type I functions [18], quasipseudo-Type I and pseudoquasi-Type I [19] to the context of complete normed linear spaces and establish the sufficiency of KKT type optimality conditions under weaker invexity assumptions than that of Brandao et al. [16]. We also obtain various duality results under aforesaid assumptions.

This paper has different sections. Section 2 consists some basic definitions and preliminary results. In section 3, we establish sufficient optimality conditions for generalized Pareto minimization problem defined on complete normed spaces. In section 4, we derive some results on weak and strong duality. Section 5 consists conclusions on our results.

2. Definitions and Preliminaries

The following definitions are from [3].

Definition 2.1 A normed space X is called a complete normed space if every Cauchy sequence in X is convergent.

Definition 2.2 A function $g : X \to R$ is said to be locally Lipschitz at $x \in X$, if and only if there exists a positive number L and a neighborhood N of x such that, for any $y, z \in N$, one has

$$|g(y) - g(z)| \leq L\|y - z\|.$$

The function g is said to be Lipschitz on X, if and only if the above condition is satisfied for all $x \in X$.

Definition 2.3 Let $g : X \to R$ be a locally Lipschitz function at $x \in X$. The Clarke generalized directional derivative of g at $x \in X$ in the direction of vector v is denoted by $g^o(x; v)$ and is defined as

$$g^o(x, v) = \lim_{\substack{y \to x \\ t \downarrow 0}} \sup \frac{g(y + tv) - g(y)}{t}.$$

Definition 2.4 Let $g : X \to R$ be a locally Lipschitz function at $x \in X$. The generalized gradient of g at $x \in X$ is denoted by $\partial g(x)$ and is defined as

$$\partial g(x) = \left\{\hat{x} \in \hat{X} : g^o(x, v) \geq \langle \hat{x}, v \rangle,\ \forall\, v \in X\right\}.$$

where $\hat{X}$ is the topological dual of X and $\langle .\,,. \rangle$ is the duality pairing.

Definition 2.5 Let $U(\neq \emptyset) \subseteq X$. The distance function $d_U(.) : X \to R$ is defined by

$$d_U(x) = \ inf\{\|x - v\| : v \in U\}$$

Definition 2.6 A vector $x \in X$ is said to be tangent to U at $u \in U$ if

$$d_U^0(u, x) = 0$$

The Clarke tangent cone of U at u is denoted by $T_U(u)$ and is defined as

$$T_U(u) = \{x \in X : d_U^0(u, x) = 0\},$$

Where d_U denotes the distance function related to U.

Definition 2.7 [16] A function $f : X \to Z$ is said to be strongly compact Lipschitzian $x \in X$ if there exist a multifunction $i : X \to comp(G)$ [where $comp(G)$ is the set of all normed compact subsets of G] and a function $r : X \times X \to R_+$ such that

$$(a)\ \lim_{\substack{y \to x \\ v \to 0}} r(y, v) = 0,$$

(b) $\exists\, \lambda > 0$ s.t. $t^{-1}\{f(y+tv) - f(y)\} \in i(d) + \|v\| r(y,t) B_Z$, $\forall\, y \in x + \lambda B_Z$ and $t \in (0, \lambda)$, where B_Z denotes the closed unit ball around the origin of Z,

(c) $i(0) = \{0\}$ and i is upper semicontinuous.

Note 2.1 If Z is finite dimensional complete normed linear space, then f is strongly compact Lipschitzian x if and only if it is locally Lipschitz near x.

If f is, then for all $\hat{z} \in \hat{Z}$, $(\hat{z} o g)(x) = \langle \hat{z}, f(x) \rangle$ is locally Lipschitz.

Let W and V are subsets of Y and Z respectively. Both subsets denote pointed closed convex cones with non-empty interior.

Let $\widehat{W}$ and $\widehat{V}$ are dual cones of W and V respectively. The cone W induces a partial order relation $\leq$ on Y defined by

$w_1 \leq w_2$ if $w_2 - w_1 \in W$

$w_1 < w_2$ if $w_2 - w_1 \in int\ W$

The negation of $w_1 \leq w_2$ is $w_1 \geq w_2$. Similarly the negation of $w_1 < w_2$ is $w_1 > w_2$. The cone V induces a partial order relation $\leq$ on Z.

Definition 2.8 [20] A locally Lipschitz real-valued function $g : X \to R$ is said to be invex function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g(w) - g(u) \geq g^0\big(u;\ \theta(w,u)\big).$$

If above inequality holds for each $u, w \in U$, then g is invex on U.

Definition 2.9 The functions $g : X \to Y$ and $h : X \to Z$ are invex, if $(\widehat{w} o g)$ and $(\widehat{v} o h)$ are invex functions, for each $\widehat{w} \in \widehat{W}$ and $\widehat{v} \in \widehat{V}$.

Definition 2.10 [17] (g, h) is said to be Type I function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g(w) - g(u) \geq g^0(u;\ \theta(w,u)),$$

$$-h(u) \geq h^0\big(u;\ \theta(w,u)\big).$$

Definition 2.11 [18] (g, h) is said to be quasi-Type I function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g(w) \leq g(u) \Longrightarrow g^0(u;\ \theta(w,u)) \leq 0,$$

$$-h(u) \leq 0 \Longrightarrow h^0\big(u;\ \theta(w,u)\big) \leq 0.$$

Definition 2.12 [18] (g, h) is said to be pseudo-Type I function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g^0(u;\ \theta(w,u)) \geq 0 \Longrightarrow g(w) \geq g(u),$$

$$h^0\big(u;\ \theta(w,u)\big) \geq 0 \Longrightarrow -h(u) \geq 0.$$

Definition 2.13 [19] (g,h) is said to be quasipseudo-Type I function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g(w) \leq g(u) \Longrightarrow g^0(u;\ \theta(w,u)) \leq 0,$$

$$h^0\big(u;\ \theta(w,u)\big) \geq 0 \Longrightarrow -h(u) \geq 0.$$

Definition 2.14 [19] (g,h) is said to be quasistrictlypseudo-Type I function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g(w) \leq g(u) \Longrightarrow g^0(u;\ \theta(w,u)) \leq 0,$$

$$h^0\big(u;\ \theta(w,u)\big) \geq 0 \Longrightarrow -h(u) > 0.$$

Definition 2.15 [19] (g,h) is said to be pseudoquasi-Type I function with respect to U at $u \in U$ if for each $w \in U$, there exists $\theta(w,u) \in T_U(u)$ such that

$$g^0(u;\ \theta(w,u)) \geq 0 \Longrightarrow g(w) \geq g(u),$$

$$-h(u) \leq 0 \Longrightarrow h^0\big(u;\ \theta(w,u)\big) \leq 0.$$

The functions $g: X \to Y$ and $h: X \to Z$ are Type I, quasi-Type I, pseudo-Type I, $\text{quasipseudo} - \text{Type}\, I$ and pseudoquasi-Type I at $u \in U$ if $(\widehat{w}og)$ and $(\widehat{v}oh)$ are Type I, quasi-Type I, pseudo-Type I, $\text{quasipseudo} - \text{Type}\, I$ and pseudoquasi-Type I respectively, for each $\widehat{w} \in \widehat{W}$ and $\widehat{v} \in \widehat{V}$.

3. Optimality Conditions

We consider the following generalized Pareto minimization problem $(GPMP)$:

$$min\{g(u): u \in U, -h(u) \in V\} \qquad (GPMP)$$

Where $g: X \to Y$ and $h: X \to Z$ are strongly compact Lipschitzian at $\bar{u} \in X$ and $U(\neq \emptyset) \subseteq X$. The subset V of Z is a pointed closed convex cone with non-empty interior.

Let S is the non-empty set of all feasible solutions of $(GPMP)$, defined by

$$S = \{u \in U: h(u) \leq 0\}$$

Definition 3.1 A feasible solution $\hat{u} \in S$ is a weak Pareto-optimal solution for $(GPMP)$, if there is no $u \in S$ such that $g(u) < g(\hat{u})$.

Proposition 3.1 [14] If $\hat{u} \in S$ is a weak Pareto-optimal solution for $(GPMP)$, then then there exists a non-zero pair of vectors $(\widehat{w}, \widehat{v}) \in \widehat{W} \times \widehat{V}$ such that, for some $\alpha > 0$,

$$0 \in \partial(\widehat{w}og + \widehat{v}oh + \alpha d_U)(\hat{u}),$$

$$\langle \widehat{v}, h(\hat{u}) \rangle = 0$$

We adopt the following Slater-type constraint qualification:

Definition 3.2 The restrictions of $(GPMP)$ satisfy the Slater condition if there exists $\bar{u} \in U$ such that $h(\bar{u}) < 0$.

Theorem 3.1 (Sufficient optimality condition) Suppose that there exist $\hat{u} \in S$ and $\hat{w} \in \widehat{W}, \hat{w} \neq 0, \hat{v} \in \hat{V}$, such that, for some $\alpha > 0$,

$$0 \in \partial(\hat{w}og + \hat{v}oh + \alpha d_U)(\hat{u}) \quad (1)$$

$$\langle \hat{v}, h(\hat{u}) \rangle = 0 \quad (2)$$

If $(\hat{w}og, \hat{v}oh)$ is Type I functionwith respect to U at $\hat{u} \in S$ then $\hat{u}$ is a weak Pareto-optimal solution for $(GPMP)$.

Proof We prove the theorem by contradiction.

Let $\hat{u}$ is not a weak Pareto-optimal solution for $(GPMP)$, then $\exists\ \bar{u} \in S$ such that $g(\bar{u}) < g(\hat{u})$.

$$i.e.\quad g(\bar{u}) - g(\hat{u}) < 0.$$

Since $\hat{w} \neq 0$, we have

$$\langle \hat{w}, g(\bar{u}) - g(\hat{u}) \rangle < 0 \quad (3)$$

If g is Type I function at $\hat{u}$ then there exists $\theta(\bar{u}, \hat{u}) \in T_U(\hat{u})$ such that

$$(\hat{w}og)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) \leq \langle \hat{w}, g(\bar{u}) - g(\hat{u}) \rangle \quad (4)$$

using (3) in (4), we get

$$(\hat{w}og)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) < 0 \quad (5)$$

If h is Type I function at $\hat{u}$ then there exists $\theta(\bar{u}, \hat{u}) \in T_U(\hat{u})$ such that

$$(\hat{v}oh)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) \leq \langle \hat{v}, -h(\hat{u}) \rangle$$

Using (2) in above inequality, we find

$$(\hat{v}oh)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) \leq 0 \quad (6)$$

From (5) and (6), we find

$$(\hat{w}og)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) + (\hat{v}oh)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) < 0 \quad (7)$$

From (1), we have

$$0 \leq (\hat{w}og)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big) + (\hat{v}oh)^0\big(\hat{u}, \theta(\bar{u}, \hat{u})\big)$$

which contradicts (7).

Hence, $\hat{u}$ is a weak Pareto-optimal solution for $(GPMP)$

Theorem 3.2 Suppose that there exist $\hat{u} \in S$ and $\hat{w} \in \widehat{W}, \hat{w} \neq 0, \hat{v} \in \hat{V}$, such that, for some $\alpha > 0$,

$$0 \in \partial(\hat{w}og + \hat{v}oh + \alpha d_U)(\hat{u}) \quad (1)$$

$$\langle \hat{v}, h(\hat{u}) \rangle = 0 \tag{2}$$

If (g, h) is pseudoquasi-Type I function with respect to U at $\hat{u}$ for the same $\theta(\bar{u}, \hat{u}) \in T_U(\hat{u})$, then $\hat{u}$ is a weak Pareto-optimal solution for $(GPMP)$.

Proof Let (g, h) is pseudoquasi-Type I function with respect to U at $\hat{u}$ for the same $\theta(\bar{u}, \hat{u}) \in T_U(\hat{u})$,then by definition, we have

$$g^0(\hat{u};\ \theta(\bar{u}, \hat{u})) \geq 0 \Longrightarrow g(\bar{u}) \geq g(\hat{u}) \tag{3}$$

$$-h(\hat{u}) \leq 0 \Longrightarrow h^0(\hat{u};\ \theta(\bar{u}, \hat{u})) \leq 0 \tag{4}$$

Since $\langle \hat{v}, h(\hat{u}) \rangle = 0$, we obtain

$$(\hat{v}oh)^0(\hat{u}, \theta(\bar{u}, \hat{u})) \leq 0 \tag{5}$$

From inequality (1), we obtain

$$(\widehat{w}og)^0(\hat{u}, \theta(\bar{u}, \hat{u})) \geq 0 \tag{6}$$

Which implies that

$$\langle \widehat{w}, g(\bar{u}) - g(\hat{u}) \rangle \geq 0 \tag{7}$$

Since $\widehat{w} \neq 0$, therefore we find the following.

$$g(\bar{u}) > g(\hat{u})$$

Hence, $\hat{u}$ is a weak Pareto-optimal solution for $(GPMP)$.

Theorem 3.3 Suppose that there exist $\hat{u} \in S$ and $\widehat{w} \in \widehat{W}, \widehat{w} \neq 0, \hat{v} \in \hat{V}$, such that, for some $\alpha > 0$,

$$0 \in \partial(\widehat{w}og + \hat{v}oh + \alpha d_U)(\hat{u}) \tag{1}$$

$$\langle \hat{v}, h(\hat{u}) \rangle = 0 \tag{2}$$

If (g, h) is quasistrictly-pseudo-Type I function with respect to U at $\hat{u}$ for the same $\theta(\bar{u}, \hat{u}) \in T_U(\hat{u})$, then $\hat{u}$ is a weak Pareto-optimal solution for $(GPMP)$.

The proof of this theorem is very easy and hence omitted.

4. Duality Result

We assume the following dual of $(GPMP)$:

$$maximize\ \ g(x), \tag{D}$$

subject to constraints: $x \in U,\ \widehat{w} \in \widehat{W},\ \widehat{w} \neq 0,\ \hat{v} \in \hat{V},\ \langle \hat{v}, h(x) \rangle \geq 0,$

$$0 \in \partial(\widehat{w}og + \hat{v}oh + \alpha d_U)(x)$$

We establish some results on weak and strong duality between problems $(GPMP)$ and (D).

Theorem 4.1 (Weak Duality) Let s and $(x, \widehat{w}, \hat{v})$ be feasible solutions for problems $(GPMP)$ and (D), respectively. If (g, h) is Type I function with respect to U at $x \in U$, for the same θ, then

$$g(s) < g(x)$$

Proof We prove the result by contradiction.

Let $\hat{s}$ and $(x, \widehat{w}, \hat{v})$ are feasible solutions for problems $(GPMP)$ and (D), respectively and $g(\hat{s}) < g(x)$.

$$i.e.\ \ g(\hat{s}) - g(x) < 0 \tag{1}$$

Since $\widehat{w} \neq 0$, we find

$$\langle \widehat{w}, g(\hat{s}) - g(x) \rangle < 0 \tag{2}$$

Since (g, h) is Type I function with respect to U at $\hat{s}$, $\exists\ \theta(\hat{s}, x) \in T_U(x)$ such that

$$(\widehat{w}og)^0\big(x, \theta(\hat{s}, x)\big) \leq \langle \widehat{w}, g(\hat{s}) - g(x) \rangle \tag{3}$$

Using (2) in (3), we find

$$(\widehat{w}og)^0\big(x, \theta(\hat{s}, x)\big) < 0 \tag{4}$$

Since $\langle \hat{v}, h(x) \rangle \geq 0$, we find

$$-\langle \hat{v}, h(x) \rangle \leq 0 \Rightarrow (\hat{v}oh)^0\big(x, \theta(\hat{s}, x)\big) \leq 0 \tag{5}$$

From (4) and (5), we find

$$(\widehat{w}og)^0\big(x, \theta(\hat{s}, x)\big) + (\hat{v}oh)^0\big(x, \theta(\hat{s}, x)\big) < 0 \tag{6}$$

From (D), we have

$$0 \in \partial(\widehat{w}og + \hat{v}oh + \alpha d_U)(x)$$

Hence, we find the following.

$$0 \leq (\widehat{w}og)^0\big(x, \theta(\hat{s}, x)\big) + (\hat{v}oh)^0\big(x, \theta(\hat{s}, x)\big) \tag{7}$$

(7) contradicts (6).

Hence, $g(s) < g(x)$. Which completes the proof.

Theorem 4.2 (Weak Duality) Let s and $(x, \widehat{w}, \hat{v})$ be feasible solutions for problems $(GPMP)$ and (D), respectively. If (g, h) is pseudoquasi-Type I function with respect to U at x, for the same θ, then

$$g(s) < g(x)$$

proof of this theorem is very easy and hence omitted.

Theorem 4.3 (Weak Duality) Let s and $(x, \widehat{w}, \hat{v})$ be feasible solutions for problems $(GPMP)$ and (D), respectively. If (g, h) is quasistrictly-pseudo-Type I function with respect to U at x, for the same θ, then

$$g(s) < g(x)$$

The proof of this theorem is very easy and hence omitted.

Theorem 4.4 (Strong Duality) If (g, h) is Type I function at all feasible points s of $(GPMP)$ with respect to U and assume that the restrictions of Problem $(GPMP)$ satisfy the Slater condition. If $\hat{u}$ is a weak Pareto-optimal solution of $(GPMP)$, then there exists $(\bar{\bar{w}}, \bar{\hat{v}}) \in \widehat{W} \times \hat{V}$ such that $\langle \bar{\hat{v}}, h(\hat{u}) \rangle = 0, (\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is a weak Pareto-optimal solution for (D), and the objective values of the problems $(GPMP)$ and (D) are equal.

Proof Since the restrictions of Problem $(GPMP)$ satisfy the Slater condition, therefore from Proposition (3.1), $\exists$ $\bar{\bar{w}}, \bar{\hat{v}}$ such that $\langle \bar{\hat{v}}, h(\hat{u}) \rangle = 0$ and $(\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is feasible for (D).

Let the feasible solution $(\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is not an optimal solution for (D), therefore there exists a feasible solution $(u, \hat{w}, \hat{v})$ for (D) such that $g(u) > g(\hat{u})$. Which is a contradiction of Theorem (4.1).

Hence $(\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is a weak Pareto-optimal solution for (D). It is obvious that the objective function values of problems $(GPMP)$ and (D) are equal at their respective weak Pareto-optimal solutions.

Theorem 4.5 (Strong Duality) If (g, h) is pseudoquasi-Type I function at all feasible points s of $(GPMP)$ with respect to U and assume that the restrictions of Problem $(GPMP)$ satisfy the Slater condition. If $\hat{u}$ is a weak Pareto-optimal solution of $(GPMP)$, then there exists $(\bar{\bar{w}}, \bar{\hat{v}}) \in \widehat{W} \times \hat{V}$ such that

$\langle \bar{\hat{v}}, h(\hat{u}) \rangle = 0, (\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is a weak Pareto-optimal solution for (D), and the objective values of the problems $(GPMP)$ and (D) are equal.

Proof Since the restrictions of Problem $(GPMP)$ satisfy the Slater condition, therefore from Proposition (3.1), $\exists$ $\bar{\bar{w}}, \bar{\hat{v}}$ such that $\langle \bar{\hat{v}}, h(\hat{u}) \rangle = 0$ and $(\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is feasible for (D).

Let the feasible solution $(\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is not an optimal solution for (D), therefore there exists a feasible solution $(u, \hat{w}, \hat{v})$ for (D) such that $g(u) > g(\hat{u})$. Which is a contradiction of Theorem (4.2).

Hence $(\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is a weak Pareto-optimal solution for (D). It is obvious that the objective function values of problems $(GPMP)$ and (D) are equal at their respective weak Pareto-optimal solutions.

Theorem 4.6 (Strong Duality) If (g, h) is quasistrictly-pseudo-Type I function at all feasible points s of $(GPMP)$ with respect to U and assume that the restrictions of Problem $(GPMP)$ satisfy the Slater condition. If $\hat{u}$ is a weak Pareto-optimal solution of $(GPMP)$, then there exists $(\bar{\bar{w}}, \bar{\hat{v}}) \in \widehat{W} \times \hat{V}$ such that

$\langle \bar{\hat{v}}, h(\hat{u}) \rangle = 0, (\hat{u}, \bar{\bar{w}}, \bar{\hat{v}})$ is a weak Pareto-optimal solution for (D), and the objective values of the problems $(GPMP)$ and (D) are equal.

Proof of this theorem is similar to that of theorem (4.4) by using weak duality theorem (4.3).

5 Conclusions

In this paper, we have obtained sufficient optimality conditions for $(GPMP)$. We have established some results on weak and strong duality between problems $(GPMP)$ and (D).

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