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ISHIKAWA ITERATIVE SCHEME TO SOLVING THE MULTIPLE-SETS SPLIT EQUALITY FIXED POINT PROBLEMS

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Abstract

In this paper we study a Ishikawa iterative scheme to solving the multiple-sets split equality fixed point problem. as well as Weak and strong convergence theorems that are proved for two countable families of λDemicontractive mappings in a Banach spaces.

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Let E be a real Banach space and E^* be a nonempty closed convex subset of B. Let $T : E^* \to E^*$ be a demicontractive map satisfying $\langle x - Tx, p\rangle \geq 0$, for all $(x, p) \in E^* \times F(T)$, where $F(T) = \{x \in E^* : Tx = x\}$. Then the Ishikawa iterative scheme defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0,1],

and converges strongly to an element of $F(T) := \{x \in E^* : Tx = x\}$.

Introduction

Let E be a real Banach space. A mapping $T : E \to E$ is said to be demicontractive if there exists a constant $k > 0$ such that

$$||Tx - p||^2 \leq ||x - p||^2 + k||x - Tx||^2 \quad (1.1)$$

for all $(x, p) \in E \times F(T)$, where $F(T) := \{x \in E : Tx = x\} \neq \emptyset$ More often than not, K is assumed to be in the interval (0, 1). However, this is a restriction of convenience. If $k = 1$, then T is called a hemicontractive map. On the otherhand, T is said to satisfy condition (A) if there exists $\lambda > 0$ such that

$$\langle x - Tx, x - p\rangle \geq \lambda||x - Tx||^2 \quad (1.2)$$

The above classes of maps were studied independently by Hicks and Kubicek [3] and Maruster [5]. It is however shown in [1] that the two classes of maps coincide if $k \in (0, 1)$ and $\lambda \in 0, \frac{1}{2}$

The class of demicontractive maps includes the class of quasi-nonexpansive and the class of strictly pseudocontractive maps. Any strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

If C is a closed convex subset of any Banach space E and $T : C \to C$ is any map, then the Ishikawa iteration sequence [2] is given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0,1], satisfying certain conditions. Several authors (see e.g [2], [3], [4], [5], [6]) have studied the convergence of the different iteration sequence to fixed points of certain mappings in certain Banach spaces. However, the Ishkawa iteration sequence [2] is very suitable for the study of convergence to fixed points of demicontractive mappings. It is well known (see e.g [1]) that demicontractivity alone is not sufficient for the convergence of the Ishkawa iteration sequence. Some additional smoothness properties of T such as continuity and demiclosedness are necessary.

A map T is said to be demiclosed at a point $x0$ if whenever $\{x_n\}$ is a sequence in the domain of T such that $\{x_n\}$ converges weakly to $y_0 \in D(T)$ and $T\{x_n\}$converges strongly to x_0, then $Tx_0 = y_0$.
In [7], Maruster studied the convergence of the Ishkawa iteration sequence for demicontractive maps, in finite dimensional spaces, with an application to the study of the so-called relaxation algorithm for the solution of a particular convex feasibility problem. More precisely, he proved the following:

Theorem 1 [7]. Let $T : \mathfrak{R}^m \to \mathfrak{R}^m$be a nonlinear mapping, where $\mathfrak{R}^m$is the m-Euclidean space. Suppose the following are satisfied:

(i) I − T is demiclosed at 0

(ii) T is demicontractive with constant k, or equivalently T satisfies condition A with $\lambda = \frac{1-k}{2}$. Then the Ishkawa iteration sequence converges to a point of $F(T)$ for any starting x_0.

Maruster [1] noted that in infinite dimensional spaces, demicontractivity and demiclosedness of T are not sufficient for strong convergence. However, the two conditions ensure weak convergence. More precisely, he proved the following: **Theorem 2 [5]** Let $T : C \to C$ be a nonlinear mapping with $F(T) \neq \emptyset$where C is a closed convex subset of a real Banach space E .Suppose the following conditions are satisfied:

(i) I − T is demiclosed at 0

(ii) T is demicontractive with constant k, or equivalently T satisfies condition with $\lambda = \frac{1-k}{2}$

(iii) $0 < a \leq \alpha_n \leq b < 2\lambda = 1 - k$

Then the Mann iteration sequence converges weakly to a fixed point of $F(T)$, for any starting x_0.

Strong Convergence

As noted above, demicontractivity and demiclosedness of T are not sufficient for strong convergence of the Mann iteration sequence in infinite dimensional spaces.Some additional conditions on T, or some modifications of the Mann iteration sequence are required for strong convergence to fixed points of demi-contractive maps. Such additional conditions or modifications have been studied by several authors (see e.g [3], [4], [5], [6], [8], [9])

There is however an interesting connection between the strong convergence of the Ishikawa iteration sequence to a fixed point of a demicontractive map, T, and the existence of a non-zero solution of a certain variational inequality. This connection was observed by Maruster [], and has been studied by several authors. More precisely, Maruster proved the following theorem:

Theorem 3 [5] Suppose T satisfies the conditions of theorem 2. If in addition there exists $0 \neq e \in E$ such that $(x - Tx, e) < 0$ for all $x \in D(T)$, then starting from a suitable x_0, the Ishikawa iteration sequence converges strongly to an element of $F(T)$.

The conditions of/and the variational inequality in theorem 3 have been used and generalized by several authors (see e.g [8], [9]). The existence of a non-zero solution to the variational inequality is sometimes gotten under very stringent conditions. In [1] remark 4, Maruster and Maruster made the following observation ”It would therefore be interesting to study more closely the existence of a non-zero solution of the variational inequality”.

The purpose of this paper is to provide a mild condition, with a reversed Maruster - type inequality, that T satisfies for the strong convergence of the Mann iteration sequence to a fixed point of a demicontractive map. The condition is embodied in the following theorem:

Theorem 4 Suppose T satisfies:

(i) The conditions of theorem 2

$(x - Tx, p) \geq 0$ for all $(x, p) \in C \times F(T)$. Then starting from a suitable x_0 , the Ishikawa iteration sequence converges strongly to an element of $F(T)$.

Proof From the proof of theorem 2 (see e.g [5]), we have

$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - \alpha_n(1 - \alpha_n - k)||x_n - Tx_n||^2$ This yields, from condition (*iii*) of theorem 2, that $||x_{n+1} - p||^2 \leq ||x_n - p||^2$ (1.3)

Let $0 \neq p \in F(T)$ Choose $x_0 \in c$ such that $\langle p, p\rangle \geq \langle x_0, p\rangle$. This implies $\langle p - x_0, p\rangle \geq 0$. Then there exists $\in_0 > 0$ such that $\langle p - x_0, p\rangle \geq \in_0 ||x_n - p||^2$

Assume $\langle p - x_n, p\rangle \geq \in_0 ||x_n - p||^2$. Then

$$\begin{aligned}\langle p - x_{n+1}, p\rangle &= \langle p - [(1 - \alpha_n)x_n + \alpha_n Ty_n], p\rangle \\ &= \langle p - x_n + \alpha_n(x_n - Ty_n), p\rangle \\ &= \langle p - x_n, p\rangle + \alpha_n\langle (x_n - Ty_n), p\rangle \\ &= \langle p - x_n, p\rangle + \alpha_n\langle (x_n - y_n), p\rangle \\ &= \langle p - x_n, p\rangle + \alpha_n\langle (x_n - [(1-\beta_n)x_n + \beta_n Tx_n)], p\rangle \\ &= \langle p - x_n, p\rangle + \alpha_n\beta_n\langle (x_n - Tx_n), p\rangle \\ &\geq \langle p - x_n, p\rangle\end{aligned}$$

Since $\langle x - Tx, p\rangle \geq 0 \forall (x,p) \in C \times F(T)$ and α_n, β_n are sequences of positive numbers so,

$$\geq \epsilon_0 ||x_n - p||^2$$

$$\geq \epsilon_0 ||x_{n+1} - p||^2 \quad \text{From (1.3)}$$

Hence for all $n > 0$, we have $\epsilon_0 ||x_{n+1} - p||^2 \leq \langle p - x_n, p\rangle$ and since $x_n \rightharpoonup p$ by theorem 2, then we have $x_n \to p$.

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