QUASI PSEUDO PROJECTIVE MODULE & QUASI PSEUDO INJECTIVITY MODULE

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Abstract

In this paper we generalize the pseudo projective and pseudo Injective module byWu and jans (1967). Characterizations of semi-simple rings by using pseudo Injectivity and that of Commutative semi-simple ring by using pseudo Injectivity have been provided (Singh and Jain 1967).It has also been provided that if every module has a pseudo projective cover then it is also has a quasi projective cover and hence a projective cover.

Key words- projective, Injective, quasi projective, quasi Injective, pseudo projective, pseudo Injective, projective cover, semi ring.

Introduction

we difine in this paper-

Lemma 1.5 – Let M be a quasi pseudo projective module and T pseudo stable sub-module of Q then M/T is quasi pseudo projective.

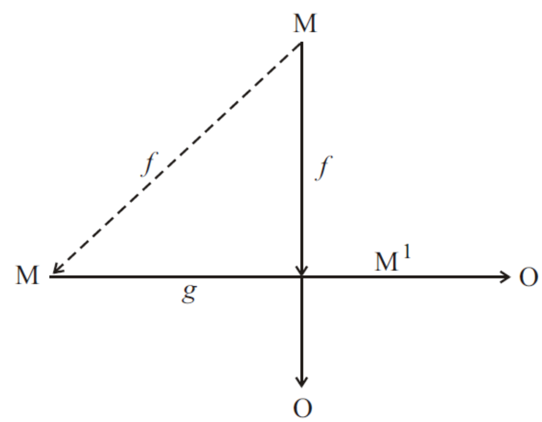
Lemma 1.6 – Quotient of a pseudo stable submodule of a quasi projective module is a pseudo stable summodule. Specifically if T is pseudo stable submodule of a quasi projective module and $A \not\subset T$ submodule, then T/A is a pseudo stable submodule of Q/A.

Lemma. 1.7- Let M be quasi pseudo projective module and let T be a quasi pseudo stable submodule of M If $C \supset T$ is not a quasi pseudo stable submodule of M then $C/T \subset M/T$ is not quasi pseudo stable.

Proposition 1.9 – If a module has a quasi projective cover, then it has a pseudo projective cover.

Theorem 1.11 – If every module has a pseudo projective cover, then it has a projective cover.

Definition 1.1- An *R*-module *M* is said to be pseudo projective if for any given *R*-module *A*, epimorphism g: $M \to M'$ and $f: M \to M'$ there exists a homomorphism $h: M \to M$ such that the following diagram is commutative, i.e. $f = g.h.$



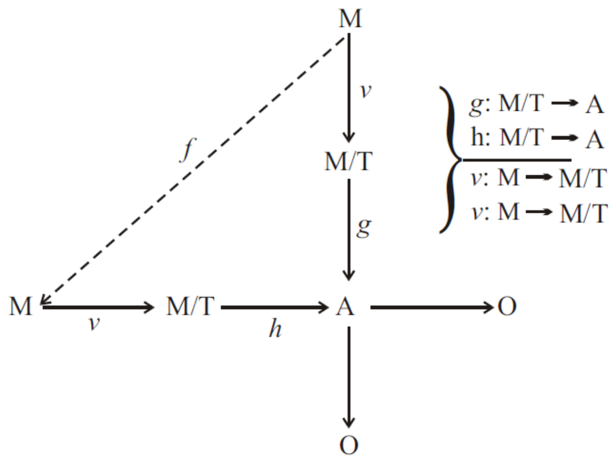
Example (1) Every projective Module has quasi pseudo module.

(2) Every M to M projective module has quasi pseudo module.

Definition : (1.2) A sub module T of Q is said to be pseudo stable if whenever for g, h : $Q \to A$ epimorphisms such that $T \subseteq \ker g \cap \ker h.$ There exists $f \in$ End Q with $g = h.f$, then $f(T) \subseteq T.$

Lemma 1.5 – Let M be a quasi pseudo projective module and T pseudo stable sub-module of Q then M/T is quasi pseudo projective.

Proof : Let g,h: $Q/T \to A$ be epimorphisms and $\nu: Q \to Q/T$ natural epimorphism, then since Q is pseudo projective there exists a homomorphism $f \in$ End (Q) such that the following diagram is commutative.



Thus $f.v.h = g.v$ (as per Def. 1.2)

$f.v.h = g.v$

Also $T \subseteq \ker g.v \cap \ker h.v.$ Hence by pseudo stability of T in $Q, f(T) \subseteq, T.$ This implies that $v.f(T) = 0,$ so that $T \subseteq \ker v.f.$

By homomorphism decomposition theorem, there exists $k \in$ End (M/T)

$v.f = k.v$

$\Rightarrow h.v.f = h.k.v$

$\Rightarrow g.v = h.k.v$

$\Rightarrow g = h.k$

So that Q/T is pseudo projective. Because $k \in$ End $M/T)$

Lemma 1.6 – Quotient of a pseudo stable submodule of a quasi projective module is a pseudo stable summodule. Specifically if T is pseudo stable submodule of a quasi projective module and $A \not\subset T$ submodule, then T/A is a pseudo stable submodule of $Q/A.$

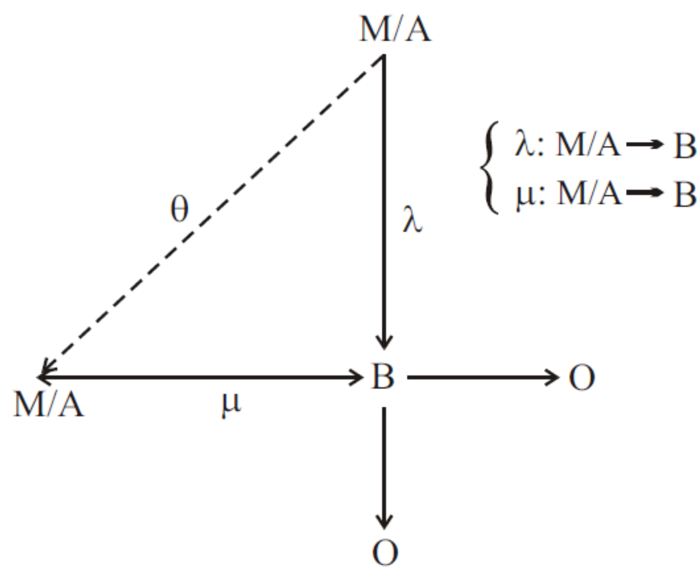
Proof: M quasi pseudo projective modules.

- T is pseudo stable submodule of M
- Quotient of pseudo stable submodule is T/A where $A \not\subset T$
- We have to proof T/A is quasi pseudo stable submodule of M/A.

(again M/A is quotient of quasi pseudo projective module)

Let - $\lambda, \mu : M/A$ be epimorphism, with $T/A \subseteq \ker \lambda \cap \ker \mu$ such that $\exists \phi \in$ End (M/A) satisfying $\Rightarrow \phi\mu = \lambda$

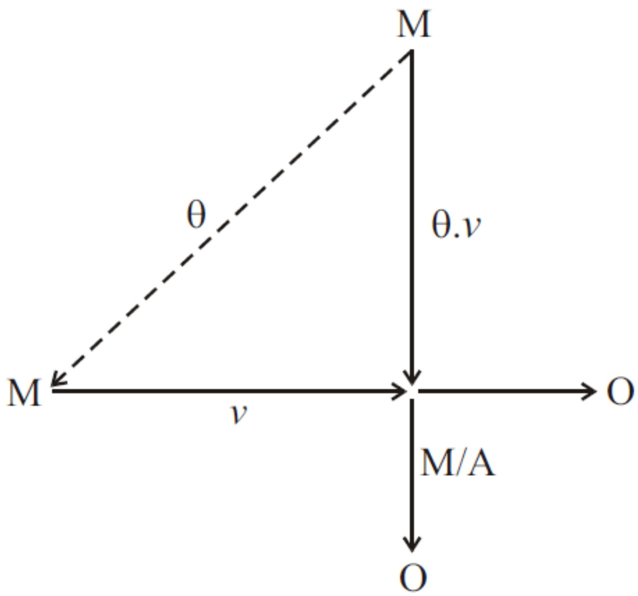
Again, Let $v : M \to M/A$ be natural pimorphism. Seince M is quasi pseudo projective module, thene existi homomorphism, $\phi \in$ End (M)



Such that $\theta.v = \phi.v$

But $\lambda = \theta.m$

So, $\lambda.v = \theta.\mu.v = \mu.\theta.v = \mu.\phi.v$ (Dig.1.1)

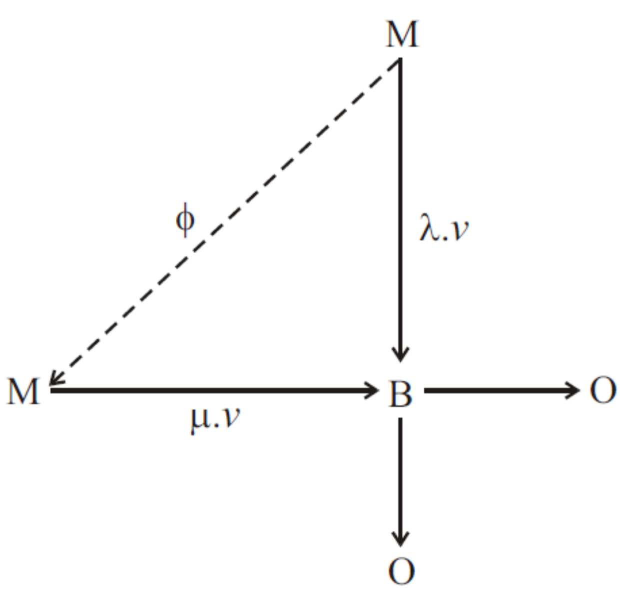


Which implies that the diagram is commutative.

Also $\lambda.v(T)=\lambda(T/A)=0$ ….(i)

and

$\mu.v(T)=\mu(T/A)=0$ ………(ii)



From (i) and (ii)

$T\subseteq \ker \lambda.v \cap \ker \mu.v$

T/A in also $\subseteq \ker \lambda \bigcap \ker \mu$

Hence $\phi(T)\subseteq T$.

If follows that, $\theta(T/A)=\theta.v(T)=v.\phi(T)\subseteq v(T)=T/A$.

Thus $T/A\subseteq Q/A$ is a pseudo stable submodule as desired.

Lemma. 1.7- Let M be quasi pseudo projective module and let T be a quasi pseudo stable submodule of M If $C\supset T$ is not a quasi pseudo stable submodule of M then $C/T\subset M/T$ is not quasi pseudo stable.

Proof: C not quasi pseudo stable in M implies that there exists $f\in$ End $(M)g,h:M\rightarrow A$ epimorphism, $C\subseteq \ker g\cap \ker h$ with $h=g.f$ such that $f(C_0)\notin C$ for some $C_0\in C$.

Let $v:M\rightarrow M/T$ natural homomorphism.

Define $\theta:M/T\rightarrow M/T$ by

$\theta(M+T)=f(M)+T$

i.e. $\theta.v=v.f$

θ is clear by a homomorphism and

$\theta(C_0+T)=f(C_0)+T\notin M/T$.

$T\subseteq \ker g\cap \ker h$ implies that there exists G and H in Hom $(M/T,(A)$,

Such that

$g=G.v$ and $h=H.v$.

Since g and h are epimorphisms, so are G and H. Now,

$h=g.f\Rightarrow H.v=G.v.f=G.\theta.v$

$\Rightarrow H=G.\theta$.

Thus there exists $\theta\in$ End (M/T) with $H=G.\theta$

$H(C/T) = H.v(C) = h(C) = 0.$

Similarly $G(C/T) = 0$

So that

$C/T \in \ker H \cap \ker G.$

But

$\theta(C_0 + T) \notin C/T.$

Hence C/T is not quasi pseudo stable.

Definition 1.8 – A quasi pseudo projective module S is said to be a pseudo projective cover of module M if –

(i) there exists a small epimorphism $\Psi : S \to M$

(ii) $\ker \Psi$ does not contain any non-zero quasi pseudo stable submodule of S.

Proposition 1.9 – If a module has a quasi projective cover, then it has a pseudo projective cover.

Proof : Let $\phi : M \to M$ be a quasi projective cover of M. Let T be a maximal pseudo stable submodule of M contained in $\ker \phi$. Then by the Lemma 1.5. M/T is pseudo projective $T \subseteq \ker \phi$ implies there exists $\Psi : M/T \to M$ such that $\phi = \Psi.v$ where $v : M \to M/T$ is natural homomorphism onto implies Ψ is onto and $\ker \Psi = \ker \phi / T$

Also

$A/T + \ker \Psi = M/T$

$\Rightarrow A \ker \phi = M$

$\Rightarrow A = M$

$\Rightarrow A/T = M/T$

$\ker \Psi \subseteq M/T$ is small submodule.

Let, if possible $C/T \subseteq \ker \Psi$ be a non-zero pseudo stable submodule of M/T.

Then $\ker \phi \supset C \supset T$. C is not pseudo stable in M since in side $\ker \phi, T$ is maximal submodule. Hence C/T is not pseudo stable by Lemma 1.7. Thus $\Psi : M/T \to M$ is a pseudo projective cover of M.

Theorem 1.11 – If every module has a pseudo projective cover, then it has a projective cover.

Proof: Consider any module M. there exists a free module F having a basis X of the same cardinality as that of M. then the one-one mapping of X onto M extends to an epimorphism $\Psi : F \to M$. By the hypothesis, $F \oplus M$ has a pseudo projective cover S with minimal epimorphism $\phi : S \to F \oplus M$. Denote the first and second projection of ϕ by p_1 and p_2 respectively. Then $S \xrightarrow{\phi} F \oplus M \xrightarrow{p_1} F$ is onto and so $S/\overline{M} \cong F$ where $\overline{M}$ is $\ker p_1.\phi$. By projectivity of $F, S = \overline{M} + T$ (direct) where $S/\overline{M} \cong T$. We can, therefore, express $S = F + \overline{M}$ (direct) after identifying T with F. Let i_1, i_2 denote the natural injections $F \to S$ and $\overline{M} \to S$ respectively. Since $p_2.\phi : S \to M$ is an epimorphism. Let $f = (p_2.\phi)/\overline{M} : \overline{M} \to M$ is as well an epimorphism. To show that $\ker f$ is small submodule of $\overline{M}$, let.

$A + \ker f = \overline{M}$

then

$A + \ker p_2.\phi = \overline{M}$

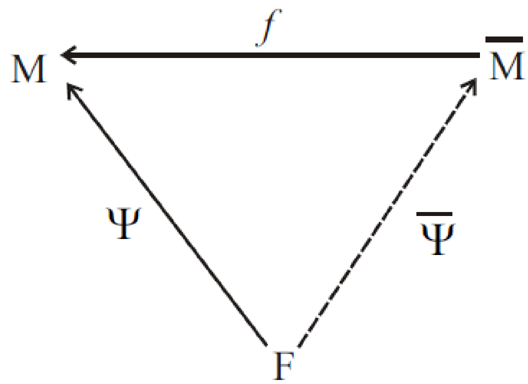
$\Rightarrow A + \ker \phi + F = F + \overline{M} = S$

$\Rightarrow A + F = F + \overline{M}$

$\Rightarrow A = \overline{M}.$

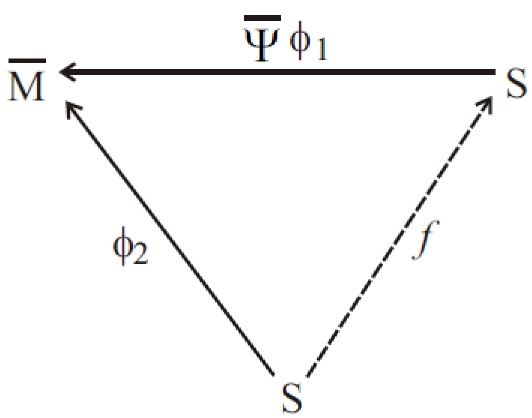
Hence $\ker f$ is a small submodule of $\overline{M}$.

Now, by projectivity of F, there exists $\Psi \in \text{Hom}\,(F, M)$ such that the following diagram is commutative, i.e. $\Psi = f.\Psi$



$\ker f \subseteq \overline{M}$ small implies Ψ is onto.

Using pseudo projectivity of $S = F + \overline{M}$,we can find homomorphism $h: S \to S$ which makes the following diagram commutative.



Where $\phi_1 = pr_1\phi$ and $\phi_2 = pr_2\phi$ are the projection mappings. Now, for any $u \in \overline{M}$,

$$u = \phi_2.i_2(u) = \Psi.\phi_1.h.i_2(u)$$

Where

$$\overline{M} \xrightarrow{\phi_1.h.i_2} F \xrightarrow{\Psi} \overline{M}$$

Is identity on $\overline{M}$

In view of the epimorphism $\Psi : F \to \overline{M}$ which is direct, $\overline{M}$ is a direct summand of F and hence it is projective. $\overline{M}$ is thus the required projective cover of M.

Corollary 1.12- If every module has a Quasi projective cover then it has a projective cover (Golan 1970, Theorem 1.1)

Proof- The result follows immediately from Propositio1.9 and 1.11

Corollary 1.13- If every module has pseudo projective cover then R is left perfect.

Corollary 1.14- If every finitely generated module has pseudo projective cover then R is left semi perfect.

Definition 1.15- A ring H is said to be left perfect if every left R-module has a projective cover. One of the several equivalent conditions characterizing left perfect rings given by Bass (1960)in Theorem P is as follows:

Every Flat module is projective.

Theorem 1.16- The following conditions are equivalent:

(1) R is left perfect.

(2) Every module has a pseudo projective cover.

(3) Every flate module pseudo projective .

Proof: (1)=> (2) : If R is left perfect, then every module has a projective cover so every module has a quasi projective cover and therefore from Proposition 1.9 every module has a pseudo projective cover.

(2)=> (1) : If every module has a pseudo projective cover then every module has a projective cover cover follows from Theorem 1.11,so R is left perfect.

(1)=> (3) : If R is left perfect then every flat module is projective from Bass (1960, Theorem).

So it is pseudo projective.

(3)=> (1) : To prove that R is left perfect , it suffice to prove that every flat module is projective.

Let M be a flat module. Let F be a free module with basis M. Then F is flat and $F \oplus M$ is flat follows from (lambek 1966,Prop.2,p.133)

Therefore $F \oplus M$ is pseudo projective. Since M is homomorphic image of F,by Lemma 1.3

M is isomorphic to a direct summand of F and hence M is projective. This completes the proof.

Pseudo Injective Modules

Theorem 2.1- For a commutative ring R the following conditions are equivalent:

(1) The direct sum of every two pseudo injective module is pseudo injective .

(2) Every pseudo injective module is injective.

(3) R is semi-simple.

This theorem is an analogue in pseudo injective case of the corresponding theorem (Koehler 1966, Theorem 2.2) given for quasi injective module. The proof in the latter case holds here also with the change pseudo injective for quasi injective.

Corollary 2.2 – For a commutative ring R the following conditions are equivalent:

(1) The direct sum of every two quasi injective modules is quasi injective.

(2) Every quasi injective module is injective.

(3) R is semi-simple.

Proof: The result follows from the above theorem and the fact that every quasi injective modules is pseudo injective.

Recall that a ring R is called a left V-ring if every simple left R-module is injective.

The following theorem is due to Villiamayor (see Faith 1966).

Theorem 2.3 –The following are equivalent :

(1) R is ring.

(2) Every left ideal of R is the intersection of maximal left ideals which contain it.

(3) For every left R-module M Rad M=0.

Remark 2.4 –In Theorem (1)=> (2) and (2)=> (1) is true without R being commutative

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