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Soft Fixed-Point Theorems in Generalized S-Metric Spaces

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Abstract

In the present research paper, some soft fixed-point theorems are proved in generalized S metric space for rational mappings. The established results satisfy well known results on fixed point theory in specified conditions.

Key Words: Soft fixed point, S- metric space, EA Property, compatible mapping

Introduction and Preliminaries:

The problems in many fields deal with uncertain data and are not successfully modelled in classical mathematics. For this purpose, the theory of soft sets initiated by Molodstov [12] and established the fundamental results of this new concept. Das and Samanta [6,7] introducing the concept of soft metric spaces which is based on soft point of soft sets. In 2012, S. Sedghi, et al.[17] introduced the notion of S-metric which is generalization of G-metric spaces [13] and D* metric [16]. Also, Sedghi and N.V. Dung [18] investigated some generalized fixed point theorems in S-metric spaces which is generalization of [17]. Aamri and Moutawakil [1] introduced E.A. property which is generalization of non-compatible mapping in metric spaces. In the present paper the notion of soft S-metric spaces which is generalization of S-metric spaces is discussed and some soft fixed point theorems are established in soft s-metric space. Obtained results are motivated by [17-18]. Throughout this paper,

X = Initial universe, A= Set of parameters , (X) = Power set of X.

Definition [Modification in 17]

$\tilde{X}$ = absolute soft set, A = set of parameters $\neq \emptyset$, $\mathbb{SE}(\tilde{X})$ = collection of all soft points of $\tilde{X}$, $\mathbb{R}(A)^*$ = set of all non-negative soft real numbers.

A mapping $S: \mathbb{SE}(\tilde{X}) \times \mathbb{SE}(\tilde{X}) \times \mathbb{SE}(\tilde{X}) \longrightarrow [0, \infty)$ is said to be a soft S-metric on $\tilde{X}$ if S satisfies the following axioms holds for all $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{a} \,\tilde{\in}\, \tilde{X}$,

(S_1) $\Theta \,\tilde{\leqq}\, S(\tilde{x}, \tilde{y}, \tilde{z})$,

(S_2) $S(\tilde{x}, \tilde{y}, \tilde{z}) = \Theta$ iff $\tilde{x} = \tilde{y} = \tilde{z}$

(S_3) $S(\tilde{x}, \tilde{y}, \tilde{z}) \,\tilde{\leqq}\, S(\tilde{x}, \tilde{x}, \tilde{a}) + S(\tilde{y}, \tilde{y}, \tilde{a}) + S(\tilde{z}, \tilde{z}, \tilde{a})$

Then, the soft set $\tilde{X}$ with a soft S-metric S on $\tilde{X}$ is called a soft S-metric space and is denoted by $(\tilde{X}, S, A)$ or $(\tilde{X}, S)$.

Let $(\tilde{X}, S)$ be soft S-metric space. If every Cauchy sequence of soft elements in $\tilde{X}$ is convergent in $\tilde{X}$, then $(\tilde{X}, S)$ is called complete soft S-metric space.

A mapping T: $\tilde{X} \longrightarrow \tilde{X}$ is called soft contraction if $S(T\tilde{x}, T\tilde{x}, T\tilde{y}) \,\tilde{\leqq}\, \rho. S(\tilde{x}, \tilde{x}. \tilde{y})$ for all $\tilde{x}, \tilde{y} \in \tilde{X}$ with $\rho \in [0,1)$.

Some remarks are to be given here which are motivated by [6-7, 12, 17]

(i) Soft S-metric $(\tilde{X}, S)$ is jointly continuous on all three variables.

(ii) Let $(\tilde{X}, S)$ be soft S-metric space $\Rightarrow S(\tilde{x}, \tilde{x}, \tilde{y}) = S(\tilde{y}, \tilde{y}, \tilde{x})$ for all $\tilde{x}, \tilde{y} \in \tilde{X}$.

(iii) In soft S-metric space. If there exists sequence $\{\widetilde{x_n}\}$ and $\{\widetilde{y_n}\}$ be the soft elements in $\tilde{X}$, such that $\lim_{n\to\infty} \widetilde{x_n} \longrightarrow \tilde{x}$ and $\lim_{n\to\infty} \widetilde{y_n} \longrightarrow \tilde{y}$, then

$$\lim_{n\to\infty} S(\widetilde{x_n}, \widetilde{x_n}, \widetilde{y_n}) = S(\tilde{x}, \tilde{x}, \tilde{y}).$$

(iv) Let $(\tilde{X}, S)$ be soft S-metric space and T: $(\tilde{X}, S) \longrightarrow (\tilde{X}, S)$ be mapping. If there exist soft elements $\widetilde{x_0} \in \tilde{X}$ such that $T\widetilde{x_0} = \widetilde{x_0}$, then the $\widetilde{x_0}$ is called soft fixed point in T.

Main Results:

Results are established for generalization of S- metric space as soft S-metric space.

Theorem 2.1: $\tilde{X}$ be a soft set. Let $(\tilde{X}, S)$ be complete soft S-metric space. F and T are compatible mapping, such that map $F, T: \tilde{X} \longrightarrow \tilde{X}$ satisfies:

(i) $F(\tilde{X}) \tilde{\subseteq} T(\tilde{X})$, F or T is continuous;

(ii)$S(F\tilde{x}, F\tilde{y}, F\tilde{z}) \leq \alpha. S(T\tilde{x}, T\tilde{y}, T\tilde{z}) + \delta \left[\frac{S(F\tilde{x},F\tilde{y},T\tilde{z})+S(T\tilde{x},T\tilde{y},F\tilde{z})}{1+S(F\tilde{x},F\tilde{y},T\tilde{z}).S(T\tilde{x},T\tilde{y},F\tilde{z})S(T\tilde{x},T\tilde{y},T\tilde{z})}\right]$ [2.1.i]

For all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \tilde{X}$ and $\alpha, \delta \geq 0$ with $\alpha + 3\delta < 1$. Then F and T has unique common soft fixed point.

Proof: For any arbitrary $\widetilde{x_0} \tilde{\in} \tilde{X}$,define a sequence $\{\widetilde{y_n}\}$ in $\tilde{X}$ such that

$$\widetilde{y_n} = F\widetilde{x_n} = T\widetilde{x_{n+1}} \qquad \text{for } n = 0,1,2, \ldots ..$$

Now to show that $\{\widetilde{y_n}\}$ is Cauchy sequence consider

$$S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) = S(F\widetilde{x_n}, F\widetilde{x_n}, F\widetilde{x_{n+1}})$$

$$S(F\widetilde{x_n}, F\widetilde{x_n}, F\widetilde{x_{n+1}}) \leq \alpha. S(T\widetilde{x_n}, T\widetilde{x_n}, T\widetilde{x_{n+1}})$$

$$+\delta\left[\frac{S(F\widetilde{x_n}, F\widetilde{x_n}, T\widetilde{x_{n+1}}) + S(T\widetilde{x_n}, T\widetilde{x_n}, F\widetilde{x_{n+1}})}{1 + S(F\widetilde{x_n}, F\widetilde{x_n}, T\widetilde{x_{n+1}}).S(T\widetilde{x_n}, T\widetilde{x_n}, F\widetilde{x_{n+1}})S(T\widetilde{x_n}, T\widetilde{x_n}, T\widetilde{x_{n+1}})}\right]$$

$$\leq \alpha. S(\widetilde{y_{n-1}}, \widetilde{y_{n-1}}, \widetilde{y_n}) + \delta\left[\frac{S(\widetilde{y_n},\widetilde{y_n},\widetilde{y_n}) + S(\widetilde{y_{n-1}},\widetilde{y_{n-1}},\widetilde{y_{n+1}})}{1+S(\widetilde{y_n},\widetilde{y_n},\widetilde{y_n}) .S(\widetilde{y_{n-1}},\widetilde{y_{n-1}},\widetilde{y_{n+1}})S(\widetilde{y_{n-1}},\widetilde{y_{n-1}},\widetilde{y_n})}\right]$$

$$\leq \alpha. S(\widetilde{y_{n-1}}, \widetilde{y_{n-1}}, \widetilde{y_n}) + 2\delta. S(\widetilde{y_{n-1}}, \widetilde{y_{n-1}}, \widetilde{y_n}) + \delta. S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}})$$

$$S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) \leq (\alpha + 2\delta)\, S(\widetilde{y_{n-1}}, \widetilde{y_{n-1}}, \widetilde{y_n}) + (\delta)\, S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}})$$

$$S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) \leq \frac{(\alpha+2\delta)}{(1-\delta)} S(\widetilde{y_{n-1}}, \widetilde{y_{n-1}}, \widetilde{y_n})$$

$$S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) \leq K. S(\widetilde{y_{n-1}}, \widetilde{y_{n-1}}, \widetilde{y_n}), \text{ Where } K = \frac{(\alpha+2\delta)}{(1-\delta)} < 1$$

Continuing this process, $S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) \leq K^n. S(\widetilde{y_0}, \widetilde{y_0}, \widetilde{y_1})$ [2.1.ii]

To prove that $\{\widetilde{y_n}\}$ is Cauchy sequence, taking $m, n \in N$ with $m > n$

$$S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_m}) \leq S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) + S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) + S(\widetilde{y_{n+1}}, \widetilde{y_{n+1}}, \widetilde{y_m})$$

$$\leq 2\, S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_{n+1}}) + S(\widetilde{y_{n+1}}, \widetilde{y_{n+1}}, \widetilde{y_m})$$

$$S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_m}) \leq 2\frac{K^n}{1-K}\, S(\widetilde{y_0}, \widetilde{y_0}, \widetilde{y_1})$$

$$\lim_{n,m\to\infty} S(\widetilde{y_n}, \widetilde{y_n}, \widetilde{y_m}) = 0$$

Therefore $\{\widetilde{y_n}\}$ is a S-Cauchy sequence in $\tilde{X}$. Since $(\tilde{X}, S)$ is complete soft S-metric space so $\exists$ $\tilde{z}$ in $\tilde{X}$ such that,

$$\lim_{n\to\infty} \widetilde{y_n} = \lim_{n\to\infty} F\widetilde{x_n} = \lim_{n\to\infty} T\widetilde{x_{n+1}} = \tilde{z}$$

Again, since F and T is continuous for definiteness $\Rightarrow T$ is continuous, therefore

$$\lim_{n\to\infty} T\widetilde{y_n} = \lim_{n\to\infty} TQ\widetilde{x_n} = \lim_{n\to\infty} TT\widetilde{x_{n+1}} = T\tilde{z}$$

Further F and T are compatible, $\Rightarrow \lim_{n\to\infty} S(FT\widetilde{x_n}, FT\widetilde{x_n}, FQ\widetilde{x_n}) = 0$

$\Rightarrow \quad \lim_{n\to\infty} FT\widetilde{x_n} = T\tilde{z}$ [2.1.iii]

$\Rightarrow \quad S(FT\widetilde{x_n}, QT\widetilde{x_n}, Q\widetilde{x_n}) \leq \alpha . S(TT\widetilde{x_n}, TT\widetilde{x_n}, T\widetilde{x_n})$

$$+\delta\left[\frac{S(FT\widetilde{x_n}, FT\widetilde{x_n}, T\widetilde{x_n}) + S(TT\widetilde{x_n}, TT\widetilde{x_n}, F\widetilde{x_n})}{1 + S(FT\widetilde{x_n}, FT\widetilde{x_n}, T\widetilde{x_n}).S(TT\widetilde{x_n}, TT\widetilde{x_n}, F\widetilde{x_n})S(TT\widetilde{x_n}, TT\widetilde{x_n}, T\widetilde{x_n})}\right]$$

limit $n \longrightarrow \infty$ $(S(T\tilde{z}, T\tilde{z}, \tilde{z})) \leq \alpha . S(T\tilde{z}, T\tilde{z}, \tilde{z})$

$$+\delta\left[\frac{S(T\tilde{z}, T\tilde{z}, \tilde{z}) + S(T\tilde{z}, T\tilde{z}, \tilde{z})}{1 + S(T\tilde{z}, T\tilde{z}, \tilde{z}).S(T\tilde{z}, T\tilde{z}, \tilde{z})S(T\tilde{z}, T\tilde{z}, \tilde{z})}\right]$$

$$\leq (\alpha)S(T\tilde{z}, T\tilde{z}, \tilde{z}) + 2\delta\left[\frac{S(T\tilde{z}, T\tilde{z}, \tilde{z})}{1 + S(T\tilde{z}, T\tilde{z}, \tilde{z}).S(T\tilde{z}, T\tilde{z}, \tilde{z})S(T\tilde{z}, T\tilde{z}, \tilde{z})}\right]$$

$$\leq (\alpha)S(T\tilde{z}, T\tilde{z}, \tilde{z}) + 2\delta\, S(T\tilde{z}, T\tilde{z}, \tilde{z})$$

$$\leq (\alpha + 2\delta)S(T\tilde{z}, T\tilde{z}, \tilde{z})$$

It is possible only if $S(T\tilde{z}, T\tilde{z}, \tilde{z}) = 0$ iff $T\tilde{z} = \tilde{z}$.

$\Rightarrow S(F\widetilde{x_n}, F\widetilde{x_n}, F\tilde{z}) \quad \leq \alpha . S(T\widetilde{x_n}, T\widetilde{x_n}, T\tilde{z})$

$$+\delta\left[\frac{S(F\widetilde{x_n}, F\widetilde{x_n}, T\tilde{z}) + S(T\widetilde{x_n}, T\widetilde{x_n}, F\tilde{z})}{1 + S(F\widetilde{x_n}, F\widetilde{x_n}, T\tilde{z}).S(T\widetilde{x_n}, T\widetilde{x_n}, F\tilde{z})S(T\widetilde{x_n}, T\widetilde{x_n}, T\tilde{z})}\right]$$

limit $n \longrightarrow \infty$ $(\quad S(\tilde{z}, \tilde{z}, Q\tilde{z}))$

$$\leq \alpha . S(\tilde{z}, \tilde{z}, \tilde{z}) + \delta\left[\frac{S(\tilde{z}, \tilde{z}, \tilde{z}) + S(\tilde{z}, \tilde{z}, Q\tilde{z})}{1 + S(\tilde{z}, \tilde{z}, \tilde{z}).S(\tilde{z}, \tilde{z}, Q\tilde{z})S(\tilde{z}, \tilde{z}, \tilde{z})}\right]$$

$$S(\tilde{z}, \tilde{z}, F\tilde{z}) \leq (\delta)S(\tilde{z}, \tilde{z}, F\tilde{z})$$

The above inequality is possible only if $S(\tilde{z}, \tilde{z}, Q\tilde{z}) = 0$ iff $F\tilde{z} = \tilde{z}$. Thus $\tilde{z}$ is common Soft fixed point of F *and* T.

Uniqueness: If $\tilde{z} \neq \widetilde{z_0}$ be two distinct common soft fixed point of F *and* T.

$$\Rightarrow S(\tilde{z}, \tilde{z}, \widetilde{z_0}) = S(F\tilde{z}, F\tilde{z}, F\widetilde{z_0})$$

$$\Rightarrow S(F\tilde{z}, F\tilde{z}, F\widetilde{z_0}) \leq \alpha . (F\tilde{z}, F\tilde{z}, F\widetilde{z_0}) + \delta\left[\frac{(F\tilde{z}, F\tilde{z}, T\widetilde{z_0}) + (T\tilde{z}, T\tilde{z}, F\widetilde{z_0})}{1 + (F\tilde{z}, F\tilde{z}, T\widetilde{z_0}).(T\tilde{z}, T\tilde{z}, F\widetilde{z_0})(F\tilde{z}, F\tilde{z}, F\widetilde{z_0}) +}\right]$$

$S(\tilde{z}, \tilde{z}, \widetilde{z_0}) \quad \leq \alpha . S(\tilde{z}, \tilde{z}, \widetilde{z_0})$

$$+\delta\left[\frac{S(\tilde{z}, \tilde{z}, \widetilde{z_0}) + S(\tilde{z}, \tilde{z}, \widetilde{z_0})}{1 + S(\tilde{z}, \tilde{z}, \widetilde{z_0}).S(\tilde{z}, \tilde{z}, \widetilde{z_0})S(\tilde{z}, \tilde{z}, \widetilde{z_0})}\right]$$

$$\leq (\alpha)S(\tilde{z}, \tilde{z}, \widetilde{z_0}) + 2\delta\left[\frac{S(\tilde{z}, \tilde{z}, \widetilde{z_0})}{1 + S(\tilde{z}, \tilde{z}, \widetilde{z_0}).S(\tilde{z}, \tilde{z}, \widetilde{z_0})S(\tilde{z}, \tilde{z}, \widetilde{z_0})}\right]$$

$$\leq (\alpha + 2\delta)S(\tilde{z}, \tilde{z}, \widetilde{z_0})$$

It is possible only if $S(\tilde{z}, \tilde{z}, \widetilde{z_0}) = 0$ iff $\tilde{z} = \widetilde{z_0}$.

$\Rightarrow$ common Soft fixed point of F *and* T is unique.

Theorem 2.2: Let $(\tilde{X}, S)$ be soft S-metric space with soft set $\tilde{X}$, $F, T: \tilde{X} \longrightarrow \tilde{X}$ be a pair of weakly compatible self-mappings satisfies:

(i) (E.A.) property satisfies by F *and* T with $T(\tilde{X})$ is closed subspace of $\tilde{X}$;

(ii) $S(F\tilde{x}, F\tilde{y}, F\tilde{z}) \leq \alpha . S(T\tilde{x}, T\tilde{y}, T\tilde{z}) + \delta\left[\frac{S(F\tilde{x}, F\tilde{y}, T\tilde{z}) + S(T\tilde{x}, T\tilde{y}, F\tilde{z})}{1 + S(F\tilde{x}, F\tilde{y}, T\tilde{z}).S(T\tilde{x}, T\tilde{y}, F\tilde{z})S(T\tilde{x}, T\tilde{y}, T\tilde{z})}\right]$ (2.2.i)

$\forall\ \tilde{x}, \tilde{y}, \tilde{z}\ \tilde{\in}\ \tilde{X}$ and $\alpha, \delta \geq 0$ with $\alpha + \delta < 1$. Then $F\ and\ T$ has unique common Soft fixed point.

Proof: It can be proved as Theorem 2.1

Theorem 2.3.: Let $(\tilde{X}, S)$ be complete soft S-metric space with soft set $\tilde{X}$. and $F, G, H : (\tilde{X}, S) \dashrightarrow (\tilde{X}, S)$ satisfying :

$$S(F\tilde{x}, G\tilde{y}, H\tilde{z}) \leq \alpha . S(\tilde{x}, \tilde{y}, \tilde{z}) + \delta \left[\frac{S(\tilde{y}, G\tilde{y}, H\tilde{z}) + S(F\tilde{x}, \tilde{y}, \tilde{y})}{1 + S(\tilde{y}, G\tilde{y}, H\tilde{z}) . S(F\tilde{x}, \tilde{y}, \tilde{y}) S(\tilde{x}, \tilde{y}, \tilde{z})} \right] \quad [2.3.i]$$

$For\ all\ \tilde{x}, \tilde{y}, \tilde{z}\ \tilde{\in}\ \tilde{X}$ with $\alpha, \delta \geq 0$, $\alpha + \delta < 1$. Then$F, G\ andH$ has unique common soft fixed point in $\tilde{X}$.

Proof: It can be proved as Theorem 2.1

Remark: All the established results can be generalized for integral type mappings motivated by [4]

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