

# SOME FIXED POINT THEOREMS IN CONE METRIC SPACE FOR POINTWISE CONTRACTIONS

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## Abstract

In this paper we give some fixed point theorems in cone metric spaces and the purpose of this paper is the investigation of some manners for finding the fixed point of pointwise contraction and asymptotic pointwise contraction mappings.

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## Introduction

Fixed point theory is one of the most important tools to solve various analytical problems in mathematics. The first known result in this direction was given by Banach known as Banach contraction principle. After the result of Banach there are so many generalizations of Banach contraction principle are presented by many mathematicians.

Recently, Guang and Xian [5] introduced the notion of cone metric spaces and proved some fixed point theorems in cone metric spaces for mappings satisfying various contractive conditions. In [6], Rezapour and Hambarani generalized some results of [5] by omitting the assumption of normality in the results.

In this present article we give some fixed point theorem in cone metric space for pointwise contraction which is generalization and extension of various known results.

Let  $E$  be a normed linear space. The subset  $P$  of  $E$  is called a cone if

- (i)  $P$  is closed, non-empty and  $P \neq \{0\}$
- (ii)  $ax + by \in P$  for all  $x, y \in P$  and all non-negative real numbers  $a, b$
- (iii)  $P \cap -P = \{0\}$

For a given cone  $P \subseteq E$ , we can define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ .  $x < y$  will stand for  $x \leq y$  and  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int}P$ , where  $\text{int}P$  denotes the interior of  $P$ . The cone  $P$  is called normal if there is a number  $M > 0$  such that for all  $x, y \in E$ ,  $0 \leq x \leq y$  implies  $\|x\| \leq M \|y\|$ .

The least positive number satisfying the above conditions is called the normal constant of  $P$ .

**Definition 1.1.** Let  $X$  be a non-empty set,  $(E, \|\cdot\|)$  a normed space that is ordered by a normal cone  $P$  with constant normal  $M = 1$ ,  $\text{int}P \neq \emptyset$  and  $\leq$  is a partial ordering with respect to  $P$ .

Also suppose that and the mapping  $d : X \times X \rightarrow E$  satisfies

(i)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,

(ii)  $d(x, y) = d(y, x)$  for all for all  $x, y \in X$

(iii)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$ , and  $(X, d)$  is called a cone metric space.

For example suppose  $E = C_R[0, 1]$ ,  $P = \{\varphi \in E : \varphi \geq 0\}$ ,  $X = R$  and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = |x - y| \varphi$  where  $\varphi : [0, 1] \rightarrow R$  such that  $\varphi(t) = e^t$ . It easy to see that  $(X, d)$  is a cone metric space.

Let  $(X, d)$  be a cone metric space,  $\{x_n\}$  a sequence in  $X$ ,  $x \in X$  and also for every  $c \in E$  with  $0 \ll c$ , there is an integer  $N > 0$  such that for all  $n > N$ ,  $d(x_n, x) \ll c$ . Then  $\{x_n\}$  is said to be convergent to  $x$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$

If for every  $c \in E$  with  $0 \ll c$  there is an integer  $N > 0$  such that for all  $n, m > N$ ,  $d(x_n, x_m) \ll c$ , Then  $\{x_n\}$  is said to be Cauchy sequence. If every Cauchy sequence is convergence, then  $X$  is called a complete cone metric space.

Also sequence  $\{x_n\}$  is said to be bounded if there is  $M \gg 0$  such that for all  $n \in N$  we have

$$d(0, x_n) \ll M.$$

Fixed point theory is an important tool for solving equations  $T(x) = x$ . However, if  $T$  does not have fixed points, then one often tries to find an element  $x$  which is in some sense closest to  $T(x)$ . A classical result in this direction is a best approximation theorem due to Ky Fan ([3]).

In this paper we consider sufficient conditions that ensure the existence of fixed point in cone metric space. It is notable that we use the results of [1-4].

## Main Results

In this section we present some fixed point theorems for pointwise contraction on cone metric spaces.

Let  $M$  be a cone metric space. A mapping  $T : M \rightarrow M$  is called a pointwise contraction if there exists a mapping  $\alpha : M \rightarrow \left[0, \frac{1}{2}\right)$  such that

$$d(Tx, Ty) \leq \alpha(x)[d(x, Ty) + d(Tx, y)] \text{ for any } x, y \in M.$$

Let  $M$  be a cone metric space. A mapping  $T: M \rightarrow M$  is called an asymptotic pointwise mapping if there exists a sequence of mappings  $\alpha_n: M \rightarrow [0,12)$  such that

$$d_T nx, T n y \leq \alpha_n dx, T ny + d_T nx, y \text{ for any } x, y \in M.$$

Now if  $\{\alpha_n\}$  converges pointwise to  $\alpha: M \rightarrow [0,12)$ , then  $T$  is called an asymptotic pointwise contraction.

Proposition 2.1. Let  $M$  be a complete cone metric space,  $P$  a normal cone with normal constant  $K$  and

$T: M \rightarrow M$  a pointwise contraction. If there is  $b \in (0,12)$  and  $n_0 \in \mathbb{N}$  such that

- 1)  $\alpha(x_n) < b$ , for all  $n > n_0$ ,  $x_0 \in M$ ,  $x_{n+1} = T^{n+1} x_0$
- 2)  $\lim_{n \rightarrow \infty} \alpha(x_n) = 0$ ;

then  $T$  has a unique fixed point in  $X$ .

Proof. Choose  $x_0 \in X$  and  $n \geq 1$ . Set

$$x_1 = T x_0, x_2 = T x_1 = T^2 x_0, \dots, x_{n+1} = T x_n = T^{n+1} x_0, \dots$$

We have

$$d_{T^n} x_n, x_{n+1} = d_T x_{n-1}, T x_n \leq \alpha_{x_{n-1}} [d_{x_{n-1}, T x_n} + d_T x_{n-1}, x_n] \leq q d(x_{n-1}, x_n)$$

where  $q = b^{-1} - b \in (0,1)$

For  $n > m > n_0$ , We have

$$d_{x_n, x_m} \leq d_{x_n, x_{n-1}} + d_{x_{n-1}, x_{n-2}} + \dots + d_{x_{m+1}, x_m} \leq q^{n-1} + \dots + q^m d(x_0, x_1),$$

Thus  $d(x_n, x_m) \leq q^{m-1} - q^m d(x_0, x_1)$ . Since  $P$  is normal we obtain,

$$d(x_n, x_m) \leq q^{m-1} - q^m K d(x_0, x_1).$$

This implies  $d(x_n, x_m) \rightarrow 0$ . Hence  $\{x_n\}$  is a Cauchy sequence. By completeness of  $M$ , there is  $x^* \in M$

such that  $x_n \rightarrow x^*$ . Thus,

$$\begin{aligned} d_T x^*, x^* &\leq d_T x_n, T x^* + d_T x_n, x^* \\ &\leq \alpha_{x_n} [d_{x_n, T x^*} + d_T x_n, x^*] + d_T x_n, x^* \\ &\leq \alpha_{x_n} d_{x_n, T x^*} + [1 + \alpha_{x_n}] d_T x_n, x^* \end{aligned}$$

$$\leq \alpha_{x_n} [d_{x_n, T x^*} + d_T x_n, T x^*] + [1 + \alpha_{x_n}] d_T x_n, x^*$$

Since  $x_n \rightarrow x^*$

Thus

$$d_T x^*, x^* \leq \alpha_{x_n} K [d_{x_n, T x^*} + d_T x_n, T x^*] + [1 + \alpha_{x_n}] K d_T x_n, x^* \rightarrow 0$$

Therefore,  $d(T x^*, x^*) = 0$ , and so  $T x^* = x^*$ . Now, if  $y^*$  is another fixedpoint of  $T$ , then

$$\begin{aligned} d_{x^*, y^*} &= d_T x^*, T y^* \leq \alpha_{x^*} [d_{x^*, T y^*} + d_T x^*, y^*] \\ &\leq 2 \alpha_{x^*} d_{x^*, T y^*} \end{aligned}$$

Hence  $d(x^*, y^*) = 0$ , and so  $x^* = y^*$ .

Therefore, the fixed point of  $T$  is .unique.

Let  $M$  be a cone metric space and  $\mathcal{F}$  a family of subsets of  $M$ . Then we say that  $\mathcal{F}$  defines a convexity structure on  $M$  if it contains the closed balls and is closed under intersection. We will say that  $\mathcal{F}$  is compact if any family  $(A_\alpha)_{\alpha \in \Gamma}$  of elements of  $\mathcal{F}$ , has a nonempty intersection provided

$\alpha \in F, A_\alpha \neq \emptyset$  for any finite subset  $F \in \Gamma$ . We will say that a function  $\phi: M \rightarrow [0, \infty)$  is  $\mathcal{F}$ -convex if  $x; \phi(x) \leq r \in \mathcal{F}$  for any  $r \geq 0$ . We will say that a convexity structure  $\mathcal{F}$  on  $M$  is  $T$ -stable if mappings

$$\phi_u := \lim_{n \rightarrow \infty} \sup d_{x_n, u} \text{ for all } u \in M;$$

are  $\mathcal{F}$ -convex .

Lemma 2.2. Let  $M$  be a cone metric space and  $\mathcal{F}$  a compact convexity structure on  $M$  which is  $T$ -stable. Then for any mappings  $\phi$  there exists  $x_0 \in M$  such that

$$\phi(x_0) = \inf \phi(x), x \in M.$$

The proof is easy and will be omitted.

Proposition 2.4. Let  $X$  be a complete cone metric space,  $P$  a normal cone with normal constant  $K$  and  $T: X \rightarrow X$ . If there is a continuous map  $\phi: P \rightarrow [0,12)$ ,  $\phi_1, \phi_2 \in \phi$  and  $n_0 \in \mathbb{N}$  such that

- 1)  $d_T x, T y \leq \phi_1 dx, y + \phi_2 dx, y + d_T x, y + d_T y, y + d(x, T x) + d(y, T y) + d(x, y)$
- 2) for any  $\phi \in \phi$  such that  $\phi_0 = 0$ .

then  $T$  has a unique fixed point in  $X$ .

Proof. Choose  $x_0 \in X$  and  $n \geq 1$ . Set

$$x_1 = T x_0, x_2 = T x_1 = T^2 x_0, \dots, x_{n+1} = T x_n = T^{n+1} x_0, \dots$$

We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T x_{n-1}, T x_n) \\ &\leq \varphi_1 d(x_{n-1}, x_n) + \varphi_2 d(x_{n-1}, T x_{n-1}) \\ &\quad + d(x_n, T x_n) \\ &\leq \varphi_1 d(x_{n-1}, x_n) + \varphi_2 d(x_{n-1}, x_n) \\ &\quad + d(x_n, T x_n) \\ &\leq \varphi_1 d(x_{n-1}, x_n) + \varphi_2 d(x_{n-1}, x_n) \\ &\quad + h d(x_{n-1}, x_n) \end{aligned}$$

where  $h = \varphi_1 + \varphi_2 + 1$

For  $n > m > n_0$ ,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + \\ &\quad d(x_{m+1}, x_m) \\ &\leq h n^{-1} + \dots + h m^{-1} d(x_0, x_1), \end{aligned}$$

thus  $d(x_n, x_m) \leq h m^{-1} - h K d(x_0, x_1)$ . Since  $P$  is normal we obtain,

$$d(x_n, x_m) \leq h m^{-1} - h K d(x_0, x_1).$$

This implies  $d(x_n, x_m) \rightarrow 0$ . Hence  $\{x_n\}$  is a Cauchy sequence. By completeness of  $M$ , there is  $x^* \in M$

such that  $x_n \rightarrow x^*$ . Thus,

$$\begin{aligned} d(T x^*, x^*) &\leq d(T x_n, T x_n) + d(T x_n, x^*) \\ &\leq \varphi_1 d(x_n, x^*) + \varphi_2 d(x_n, T x_n) \end{aligned}$$

$$d(x^*, T x^*) \leq d(x_n, x^*) + d(T x_n, x^*).$$

$$d(T x^*, x^*) \leq 0$$

which contradiction. so we have

$$d(T x^*, x^*) = 0.$$

Therefore ,

$d(T x^*, x^*) = 0$ , and so  $T x^* = x^*$ . Now, if  $y^*$  is another fixed point of  $T$ , then

$$\begin{aligned} d(x^*, y^*) &= d(T x^*, T y^*) \leq \varphi_1 d(x^*, y^*) + \varphi_2 d(x^*, T x^*) \\ &\quad + d(y^*, T y^*) \end{aligned}$$

$$\leq \varphi_1 d(x^*, y^*) + \varphi_2 d(x^*, T x^*) + d(y^*, T y^*)$$

Hence  $d(x^*, y^*) = 0$ , and so  $x^* = y^*$ . Therefore, the fixed point of  $T$  is unique.

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