International Journal of Advanced Technology & Engineering Research (IJATER)
www.ijater.com

(M is M) W -Projective Modules and dimensions

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Abstract :- In this paper, we generalize the notion of projective, injective, and flat modules and dimension. Hence, we introduce and study the notion of quasi- w projective modules and dimension.

Introduction :- Throughout this paper all rings are associative and all modules if not specified otherwise are left and unitary. Let R is a ring and M be an R- module As usual we use Pd_R (M) , id_R (M) and fd_R (M) to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. We use also gldim (R) and Wdim (R) to denote respectively, the classical global and quasi w-dimension of R. The character module, Hom_z (M, Q/2) is denote by M^1.

Recall that a ring R is called left perfect if every flat module is projective. Example of Perfect rings include quasi - Frobenius rings, that is right or left self injective rings and and right or left Artinian. It is shown that a ring is quasi - Frobenius if and only if every projective module is injective if and only if every injective module is projective. We introduce and study a new generalization of projective and injective modules and dimension.

The relation between the quasi - w projective dimension and other dimension are discussed.

Definition - 1.1:- For an R- module M, the quasi w- projective dimension of M, $qwpd_R$ (M), is defined to be the smallest integer n≥o such thatExt_R^{n+1} (m, m) = 0 for all flat modules m. If no such integer exists, set $qwpd_R$ (M) = ∞. If $qwpd_R$ (M) = 0 then M will be called a quasi w- projective module.

Example 1.2 consider the local quasi - Frobenius ring R=== $\frac{M[Z]}{Z^2}$ where m is a field, and denoted by z the residue class in R of z. Then z is a quasi- w projective R- module which is not projective.

Proof :- Since R is quasi - Frobenius, every projective (and every flat since R is perfect) Modules m is injective. The Ext. $^{i}_{R}$ ((z , m)) = 0 : i > 0.

Thus $\overline{z}$ is a quasi- w- projective R- module. Now, if we suppose that $\overline{z}$ is projective. Then, it must be free since R is local, a contradiction since $\overline{z}^2 = 0$. So, we conclude that $\overline{z}$ is not projective, as desired.

In [4] the authors defined and studied a refinement of flat modules which they called the IF modules Recall that on R-module M is said If module if Tor $^{i}_{R}$ I, M) = 0 for all right injective R- module I and all i > 0.

Proposition 1.3:- Let R be a right coherent ring. Then every quasi we project R- module is an IF R- module.

Proof :- Let m be a quasi - w projective R - module. Let E be an injective right R- module. Then $\overline{E}$ is flat. Then Ext. $^{i}_{R}$ (M,$\overline{E}$) = 0 for all i>0. While

Ext. (M,E) = $\overline{(\text{Tor}^{i}_{R} (E, \overline{M}))}$. Hence Tor $^{i}_{R}$ E, M) = 0

Thus M is an If - module.

Proposition 1.4:- Let M be a quasi- w projective R - module,

Then,

(i) Ext. (M, M*) = 0 for all I>0 ans. all M* with finite flat dimension.

(ii) Either M is projective or fd_R (M) = ∞

Proof. (i) Since Ext. $^{i}_{R}$ (M, M*) = 0 for all flat modules M* and all i > 0, the proof is immediate by dimension shifting.

(ii) Suppose that fd_R (M) < ∞ and pick a short exact sequence

$0 \rightarrow M^* \rightarrow P \rightarrow M \rightarrow 0$ where P is projective. clearly

fd_R (M*) < ∞. Then Ext. $^{i}_{R}$ (M, M*) = 0

Thus the short exact sequence splits, and so M is isomorphic to a direct summand of P and then projective.

Corollary 1.5 :- A module M is quasi - projective if and only if it is flat ans. quasi-w projective.

Proof:- Let M be an R-module. The cotorsion dimension of M, cd_R (M) is smallest integer n such that

$Ext_R^i(M, \overline{M}) = 0$ for all flat module $\overline{M}$.

The left cotorsion dimension of the R, Cot. D (R) is the supremum of cotorsion dimension of R module. If is shown in [5, corollary 7.2.6] that

l. cot. D(R) = Sup. { qpd_R ($\overline{M}$/M- Flat}.

Proposition 1.6 :- Let M be an R- module and consider the following condition.

(i) M is a quasi- w projective module.

(ii) Ext_R^i (M, P) = 0 for all i>0 and the projective modules P.

(iii) Ext_R^i (M, P) = 0 all i>0 ans. all module P with finite projective dimension.

then (i) => (ii) <=> (iii) All statements are equivalent if M is finitely presented or if l. cot. D (R) < ∞.

Proof. (i) => (ii) It is trivial

(ii) <=> (iii) Result by dimension shifting.

Let $\overline{M}$ be a flat module. By Lazard's theorem [2, section 1 N. 6 theorem 1], there is a direct system $(Li)_{i \varepsilon I}$ of finitely generated free R- modules such that $\varinjlim h_i \approx \overline{M}$

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If M is finitely presented, from [2, Exercise- 3 , P-187]

We have Ext. $(M, \overline{M}) \approx \varinjlim$ Ext. (M, Li).

Thus

in this case the implication. (iii) => (i) holds.

If l cot. D (R) $< \infty$ then qpd_R (M) $< \infty$.

Hence, in this case also the implication (iii) => (i) holds.

Proposition 1.7 :- The following statements equivalent.

(i) R is left perfect.

(ii) Every flat module is quasi w- projective.

In particular if the class of all quasi- w projective modules are closed under direct limits, then R is left perfect.

Proof. :- If R is left perfect it is clear then every flat module is quasi - w projective. As to the converse, let $\overline{M}$ be a flat module. By (i) it is quasi-w projective ans. so projective by proposition 2.4. Then R is left perfect. If the class of all quasi-w projective module is closed under direct limits, then any direct limit of projective modules is both flat and quasi-w project.

(Since every projective module is both flat and quasi-w projective). Then by corollary 1.5 every direct limit of projective module is projective.

Thus R is left perfect.

Proposition 1.8:- The following are equivalent.

(i) Every R- module is quasi-w projective.

(ii) R is quasi- Frobenius.

Proof:- This follows the fact that a ring is quasi- Frobenius if and only if every projective module is injective and that is quasi - Frobenius rings are perfect [].

A left (right) R- module M is said FP- injective if Ext. $(M,\overline{M}) = 0$ for for every finitely presented left (right) R- module M.

A ring R is said to be FC if it is left any right coherent and left right self FP- injective.

Proposition 1.9: The following are equivalent.

(i) R is FC.

(ii) Every finitely presented (left and right) module is quasi w- projective.

Proof:- Let M be a finitely presented right (or left) module any $\overline{M}$ be be a flat right (or left) module.

Then $\overline{M}$ is FP- injective by [8, lemma 4.1], so, $Ext_R^i(M, \overline{M}) = 0$ for all i>0. Thus M is quasi w- projective.

As to converse for any finitely presented right of left module M we have $Ext._R^i$

(M, R)= 0 for all i>0 by (2).

Thus, R is self right and left FP- injective.

Proposition 1.10:- For any R-module M and any positive.integer n,

the following assertions are equivalent.

(i) $qwpd_R(M) \leq n$

(ii) $Ext._R^i(M, \overline{M}) = 0$ for all i>n, all R- module M with finite flat dimension.

(iii) If $o \to G_n \to G_{n-1} \to \to G_0 \to M \to 0$

is an exact sequence of modules with G_o --------- G_{n-1} are quasi - w projective module, then G_n is a quasi w- projective

Proof:- (i) <=>(iii) the proof is by std homological algebra (ii) => (i) obvious.

(i) => (ii) Set P = qqpd$_R$ (M). by induction on m=fd$_R$ ($\overline{M}$) we pure that Ext. ^i_R (M,$\overline{M}$)=0 for all i>P. The induction stant is given by (i). If m>0 pick the short exact sequence $0 \to \overline{M}^1 \to P \to \overline{M} \to 0$ where P is a projective module. Clearly, fd^R ($\overline{M}^1$) = M-1. Thus, Ext. ^i_R = M 0 for all i>n. From the long exact sequence.

$\to$Ext. ^i_R (M,P) $\to$ Ext. ^i_R (M,$\overline{M}$) $\to$ Ext. $^{i+1}_R$ (M,$\overline{M}^1$) $\to$...........

It is clear that Ext. ^i_R (M,$\overline{M}$) = 0 for all i>n.

<u>Proposition 1.11:-</u> For any R- module M, qwpd$_R$ (M) $\leq$ Pd$_R$ (M), with equality if fd$_R$ (M) is finite.

Proof:- The first inequality follows from the fact that every projective module is quasi - w projective.

Now, set qwpd$_R$ (M) = n $<\infty$ and consider on n-step projective resolution of M as follows.

$0 \to M^1 \to P_{n-1} \to \to P_1 \to P_0 \to M \to 0$ where

all Pi are projective. Clearly, M' is quasi w- projective. If fd$_R$ (M) $< \infty$ and than if is projective by proposition 1.4. Hence Pd$_R$ (M) $\leq$n, and so the equality holds.

[2] Quasi W- projective dimension of rings:

Definition- 2.1:- The left quasi-w projective dimension of a ring R, l, qwpd (R) is defined by setting l. qwpd (R) = Sup { qwpd (M) /Mis a (left) R-Module}

Theorem 2.2 Let R be a ring and ne be a positive integer. The following are equivalent.

(i) l. qwpf (R) ≤ n.

(ii) qwpf (R/I) ≤ n for every (left ideal I or R.

(iii) id_R ($\overline{M}$) ≤ n for all flat module $\overline{M}$.

(iv) id_R (P) ≤ n for all projective module P.

Proof. (i) => (ii) and (iii) => (iv) are obvious.

(ii) => (iii) Let $\overline{M}$ be a flat module. Since qwpd R (R/I) ≤n for every iedal I of R, we have Ext. $^{i}_{R}$ (R/I, $\overline{M}$) = 0 for all i>n. Thus using the baer criterion ([7, Lemma 9.11]), idR ($\overline{M}$) ≤n.

(iv) => (i) Let M be an arbitrary module. Since

Id_R (P) ≤n for each projective module O, we have Ext. $^{i}_{R}$ (M, P) = 0 for all > n and projective module P. By dimension shifting we get $^{i}_{R}$ Ext. (M, P) = 0 for all i>n and all module P with finite projective dimension.

By [5, theorem 7.2,5] l. Cot. D (R) ≤Sup {id_R (M)/P Projective} ≤n. Thus given a flat module $\overline{M}$, we have qpd_R $\overline{(M)}$, we have qpd_R ($\overline{M}$) < ∞. Hence, Ext. $^{i}_{R}$ (M, $\overline{M}$) = 0 for all is >n. Consequently, $qwpd_R$ (M) ≤n.

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