International Journal of Advanced Technology & Engineering Research (IJATER)
www.ijater.com

Structure of Strongly Pure - Projective Module

Sneha Joshi[1,*] & Dr. M.R.Aloney[2]
1 .Research scholar-Bhagwant University,Ajmer,Rajasthan,India-305004
2. Guide: Professor, Department of mathematics,Bhagwant University,Ajmer, Rajasthan,India-305004
*Email:sneha.guha77@gmail.com

Abstract: In this paper we investigate the structure of strongly pure syzgy modules in a strongly pure projective resolution of any right R- module over an association ring R with identity element. We show that a right R- module M is strongly Pure projective if any only if there exists an integer in ≥ O and a strongly pure exact sequence O → M^J → P_n → P_{n-1} .... P_O → M^J → O with strongly pure- projective modules $P_{n'}$ P_{n-1} ...., P_O. As a consequence we get the following version of a result in Benson and Goodearl 200:

A strongly that module M^J is projective if M^J admits an exact sequence O → M^J → F_n → F_{n-1}............ → F_O → M^J → O with Projective module $F_{n'}$ $F_{n-1'}$..............F_o.

Introduction

Throughout this paper R is an association ring with an identity we denote by mod(R) the category of all right R- modules. We recall (12) that an exact sequence ....X_{n-1} → X_n → X_{n+1} → ....in Mod (R) is said to be pure [6] if the induces sequence ....X_{n-1} ⊗ L → X_n $\otimes_R$ L → X_{n+1} $\otimes_R$ L → ....of abelian group is exact for any left R-module L. An epimorphism F: Y → Z in Mod (R) is said to be pure if the exact sequence O → Kerf → Y → Z → O is pure. A submodule X of right R- module Y is said to be pure if the exact sequence O → X → Y → Y/X → O is pure. A module R is mod (R) is said to be pure- projective if for any pure epimorphism f: Y f: Y→ Z in mod (R) the induced group homomorphism HOM_R (p,f,): Hom_R (p,v,) → Hom_R (p,z) is surjective. The following facts are woll-known (14), (15), (29), (30):

(i) A module P is mod (R) is pure projective if and only if P is a direct summand of a direct sum of finitely presented modules.

(ii) Every module M is Mod (R) admits a pure-projective pure resolution P in Mod (R) that is there is a pure exact sequence.

$$\dots\dots\dots \longrightarrow P_n \xrightarrow{dn} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow O$$

where the module, $P_o \dots\dots\dots . P_n \dots\dots$ are pure projective. The main results of the paper are the following.

[1] Preliminaries on the strongly pure-projective dimension Given right R-modules M and N the n-th strongly pure extension group $Pext_R^n(M, N)$ is defined to be the n-th Cohomology group of the complex $HOM_R(P., N)$, where $P.$ is a strongly pure projective resolution of M in mod (R).

The strongly pure-projective dimension P. pdM of M^J is defined to be minimal integer m≥o (or infinity) such that $Pext_R^n(M^J -) = O$. The right strongly pure global dimension r.p. gl. Dim R of R is defined to be the minimal integer n≥o (or infinity) such that $Pext_R^n = O$. We call the ring R right strongly pure semisimple if r.p.gl.dim R = O.

Throughout this paper we denote by v an infinite cardinal number and v_o the cardinality of a countable set. A right R-module M is said to be v-generated if it is generated by a set of cardinality v and M^J is v-generated and for any epimorphism f: L $\longrightarrow$ M^J with v-generated module L the kernel kerf is v-generated or equivalently. M^J is a limit of a direct system $\{M_i^J, h_{ij}\}$ of cardinality v consisting of finitely presented modules M_i. We say that M^J is an v-directed union of submodules M_i , $i \in I$, if for each subset I_o of I of cardinality v there exists $i_O \in I$ such that $M_i \subseteq M_{io}$ for all t $\in I_o$. $\left[M^J \bigoplus_{j=1}^{n} m \right]$

$A\ Union\ U_{\xi-\lambda} M_\xi^J\ of\ sub\ module\ M_\xi^J\ of\ M\ is\ well-ordered$ and untionous if γ is an ordinal number $M_o = (O), M_\xi \subseteq M_\eta \; for$

$$\xi < \eta < \gamma \; and \; M_T = \underset{\xi - \lambda}{\cup} M_\xi^J \text{ for any limit oridnal number } T \leq \gamma.$$

Theorem 1.1 Assume that ρ is a strongly Pure Projective right R-module and let K is a submodule of P. The following condition's are equivalent.

(i) K is a strongly Pure Submodule P.

(ii) For any finitely generated Submodule X of K there exists an R-homomorphism $\psi: K$such that $I_M \Psi$ is contained in a finitely generated R-submodule of K and $\frac{\Psi}{X} = idx$.

(iii) For any finitely generated sub-module X of K there exists an R-homomorphism $\Psi: P \longrightarrow K$ such that $\frac{\Psi}{X} = idx$.

Proof: Since the module P is strongly pure-projective, there exists a module P' such that $P \oplus P'$ is a direct sum of finitely presented modules. Assume that K is a Submodule of P and let $\phi : K \longrightarrow P$ be the embedding.

(i) $\Longrightarrow$ (ii) Assume that $\phi: K \longrightarrow P$ is a strongly pure monomer phism and X is a finitely generated submodule of K. Then the monomorphism $(\phi, o): K \longrightarrow P \oplus P'$ is strongly pure and there exists a finitely. Presented direct sum M and L of $P \oplus P'$such that $(\phi, o)(x) \subseteq L.$ consider the commutative diagram.

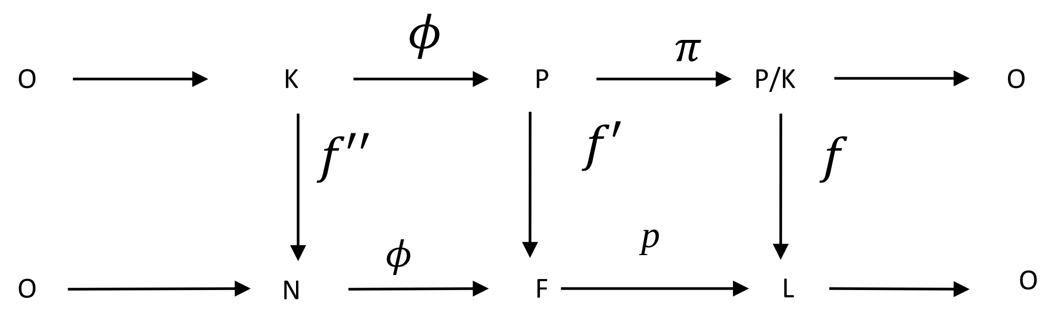
$$\begin{array}{ccccccccc}
O & \longrightarrow & K & \xrightarrow{(\phi,o)} & (P \oplus P') & \xrightarrow{\pi} & (co \ker(\phi, o)) & \longrightarrow & O \\
 & & \uparrow h' & & \uparrow h & & \uparrow h'' & & \\
O & \longrightarrow & K & \xrightarrow{\phi'} & (P \oplus P') & \xrightarrow{P} & \bar{L} & \longrightarrow & O
\end{array}$$

With exact rows, where h' is the embedding of X inot K, h is a direct sum and embedding, π is a strongly pure epimorphism and the module I is finitely presented.

It follows that there exists $\xi \in Hom_R(\bar{L}, P \oplus \bar{P})$ such that $\pi\xi = h"$, and consequently there exists $\xi \in Hom_R(L, K)$ such that $\xi'\phi' = h'$. Let $\Psi': P \oplus P' \longrightarrow K$ is an extension of ξ' to $P \oplus P'$ such that $\xi' = \psi' h$ and $I_m\psi'$ is finitely generated. Let $\Psi: P \longrightarrow K$ be the restriction of Ψ' to P. It follows that $I_M\psi'$ is contained in the finitely generated R-module $I_M\psi'$ of K and for any $x \in X$, we have $x = h'(n) = \xi u'(x) = \psi' h u'(x) = \psi'(h'(x)) = \psi'(x, o) = \psi(x)$. This shows that $\psi|X = idx$ and (ii) follows.

(ii)=> (iii) it is obvious

(iii) => (i) Assume that, for any finitely generated sub-module X of K, there exists and R-homorphism ψ: $\longrightarrow$ P K such that $\psi/$X = idx. we shall prove that K is a strongly pure submoudle of P by showing that the canonical epimorphism π:P/K is strongly pure. Let *f*:L→ P/K be a homomorphism from a finitely presented module L to P/K. Then L =? F/N, where F is a finitely generated free module and N is a finitely generated submodule of F. It is clear that exists a commutative diagram.



With exact rows, where S is the canonical epimorphism and ∅ is the canonical embedding. Then X=,'"(N) is a finitely generated submodule of K and according to our assumption, there exists an R-homomorphism ψ:P→K such that ψ/X = idx.

Note that the homomorphism $€'=\psi F'$: →K satisfies the equality f"=??. It follows that there exists €"ϵ HOM$_R$ (L,P) such that π€"F.

This shows that ? is a strongly pure epinmorphism.

Let p be a strongly pure-projective right R-module and let be a strongly pure sub-module of P. we define strongly pure-closure L^J of any R-Submodule L of K as follws, set L_0= L and fix a set L' of generators of L. for any finite subset π of L'we find a R-homomorphism ψλ:P→K $I_{M\psi\lambda}$ is contained in a finitely generated R-sub module Kλ Of K, and ψλ/λ= idλ. Let L, be the R-submodule of L generated by the set L" =???, where ? runs over all finite subset of L'. It is clear that L = L? ? L? and for any finitely generated submodule X.

Lo = L, there exists an R-homomorphism ω:P→L_1 suchthat I $_M$ωis contained in a finitely generated R-module of L_1 and ω/x = idx. By choosing a set K: of generated of L_1 and appliying the procefing above with L' and L; interchanged, we construct a submodule L_2 containing L_1 such that for any finitely generated submoduleX such that I $_M$ωis is contained in a finitely generated R-submodule of L_2 and ω/x = idx. containuing this why we detine an ascenfing sequence.

L= $L_0 \subseteq L_1 \subseteq L_2 \subseteq$ …….. $\subseteq L_W \subseteq L_{M+1} \subseteq$ ……of R-Sub modules of K, and sets $L_{0'}$, $L_{1'}$, $L_{2'}$………… $L_{M'}$, $L_{M+1'}$…….. of their generator's in such a way that for each M ≥ 0 and for any finitely generated submodule X of $L_{M'}$, there exists an R-homomorphism ω: P→ L_{M+1} such that I $_m$ω is Contained in a finitely generated R-module of L_{M+1} and ω/x=idx.

L^J= ? L_M of K is a strongly pure submodule of P and we call it a strongly pure-closure of R-submodule L of K. It is clear that L^J is not determined it uniquely by L and depends on the choice of the modules K?, set $L_{0'}$, $L_{1'}$, $L_{2'}$............ $L_{M'}$, $L_{M+1'}$......... and the R-homomorphism ψ_λ:P⟶. However, v if is an infinite cardinal number and the module L is v –generated then the sets $L_{0'}$, $L_{1'}$, $L_{2'}$............ $L_{M'}$, $L_{M+1'}$......... can be chosen of cardinality v and we get the following result.

Theorem: Lemma-2.1:, Assume that P is a strongly pure projective right R-module, K is a strongly pure sub-module of P and L is an v –generated submodule of K, where v is an infinite cardinal number. Then there exish an v – generated submodule L^J of K such that L ⊆ L^J and L^J is a strongly submodule of P.

Lemma- 2.2: Assume that v is an infinite cardinal number h:P⟶ K is a strongly pure epimorphism in mod-R,P is an v –generated strongly pure projective module and K is a strongly pure submodule of a strongly pure projective module.

(i) The module K has a directed summary from $K = \bigoplus_{\lambda \in \Omega} K_\lambda$ where Ω is a set of cardinality $\leq v$ and $K_\lambda$is a countably generated strongly pure projective submodule of K, of each $\lambda \in \Omega$.

(ii) The module kerh is *v*-generated.

Proof: Let *h:P* ⟶ *K* be a strongly pure-epimorphism. We set L=kerh and assume that the module p is *v*-generated. Then there exists a set Ω of cardinality $\leq v$ and a family of finitely generated submodules P_λ of P, with $\lambda \in \Omega$ such that $P = \bigoplus_{\lambda \in \Omega} P_\lambda$ is a directed summand. By our assumption, K is a strongly pure submodule of a strongly pure-projective module P_o.

Let P_O' be a right R-module such that $P_O \oplus P_o'$ is a direct sum of finitely presented modules. For each $\in \Omega$, we consider the commutative diagram.

$O \quad L \cap P_\lambda \quad P_\lambda \quad \bar{P_\lambda} \quad O$

$O \quad L \quad P \quad K \quad O$

With exact rows, where $\bar{P_\lambda} = P_\lambda | L \cap P_\lambda, \phi_\lambda, \phi'_\lambda, \phi''_\lambda, \xi$ are the embedding and $\gamma_\lambda$is the natural R-module homomorphism induced by ϕ''_λ. Since $V_\lambda = I_m \gamma_\lambda = h(P_\lambda)$ is a finitely generated submodule of K and K is a strongly pure submodule $V_\lambda^J$of $P_o \oplus P_o'$contained in K and containing V_λ. It follows that $V_\lambda^J$is a strongly pure sum module of an v_o generated direct summand P' of $P_o \oplus P_o'$. Then the module $P' / V_\lambda^J \leq 1$.

If the following that the submodule $V_\lambda^J$of K is strongly pure-projective. It we set $K_\lambda = V_\lambda^J$ Then obviously $K = \bigoplus_{\lambda \in \Omega} K_\lambda$is a direct summand and $K_\lambda$is a countably generated strongly pure projective sub-module of K for each $\lambda \in \Omega$.

(iii) Since the epimorphism $h: P \longrightarrow K$is strongly pure, the embedding $w_\lambda: V_\lambda^J \longrightarrow K$ extends to an R-module homomorphism $f_\lambda: V_\lambda^J \longrightarrow P$ such that $hf_\lambda = \varpi_\lambda$. Then the composed R-module homomorphism $\psi_\lambda = f_\lambda \gamma_\lambda: \bar{P_\lambda} \longrightarrow P$ satisfies $h\Psi_\lambda = \gamma_\lambda$ and by the commutativity of above diagram, there exists an R-module homomorphism $\psi_\lambda: P_\lambda \longrightarrow L$ such that $\psi_\lambda \phi_\lambda = \phi_\lambda'$. Hence we easily conclude that

$$L = \sum_{\lambda \epsilon \Omega} I_m \Psi_\lambda$$

and therefore L is v- generated, because $|\ \Omega\ | \le v$ and $I_m\psi_\lambda$ is finitely generated for and $\lambda \epsilon\ \Omega$.

(3) A strongly pure-projective structure of pure-syzygy modules

The aim of this section, the strongly pure-pronective structure of then-th strongly pure-syzygy module of any right R-module M, that is that is the strongly puresubmodule Ker d_n *of* P_n in a strongly pure-projective resolution of $M^{J.}$

Proposition: 3.1 : Assume that R is a ring, v is an infinite cardinal number, M^{J} is a right R-module, $n \ge o$ an integer and

- $O \longrightarrow K_n \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots\ldots\ldots . \longrightarrow P \xrightarrow{d_n} P_O \xrightarrow{d_o} M^J \longrightarrow O$

Is a strongly pure exact sequence, where $K_n = ker\ d_n$ and the module $P_o, P_1, \ldots\ldots P_n$ are strongly pure-projective.

(i) For any v-generated sub module N of K_n and any v- generated submodule L of $K_O = ker\ d_o$ there exist an v- generated strongly pure submodule N^{Jn} of P_n and v– generated strongly pure submodule $L^{Jo}$of P_o an v- generated direct summands $P_1', P_2', P_3' \ldots\ldots\ldots P_o'$ of $P_1, P_2 \ldots\ldots .. P_n$, respectively, such that $d_J\left(P_J'\right) \subseteq P_{J-1}'$for $j = 1,2\ \ldots n, N =\subseteq N^{Jn} \subseteq K_n = \ker d_o$ and for each $n \ge 1$, the sequence.

- $O \longrightarrow N^{Jn} \longrightarrow P_n' \xrightarrow{d_n'} P'_{n-1} \xrightarrow{d'_{n-1}} \ldots\ldots\ldots . \xrightarrow{d_2'} P_1' \xrightarrow{d_1'\ o} L^{Jn} \longrightarrow O$ is strongly pure exact, where d_J' is the restriction of $d_j$to P_j'. In case n=o we have $N^{Jo} = L^{Jo}$.

Proof: (i) since any strongly pure-projective module is a direct summand of a direct summand of finitely presented modules then according to the well-known [13] there are pairwise disjoint sets $I_0, I_1, I_2 \ldots\ldots .. I_n$and countably generated strongly pure-projective modules Q_t, with $t \epsilon I_0 \cup I_1 \cup I_2 \cup\ \ldots\ldots\ldots\ldots . \cup I_n$such that, for each for

$j \in \{0,1,2 \ldots \ldots . n\}$ the strongly pure-projective module P_j in (*) has the form.

$$P(I_j) = \bigoplus_{t \in \mathrm{I}} Q_t$$

Up to isomorphism without loss of generality we can suppose that $P_j = P(I_j)$ for $j=0,1 \ldots \ldots \ldots . n$.

Assume that for each $j \in \{0,1,2 \ldots . n\}$, the strongly pure-projective module P_n in (*) has the form $P(I_j)$ as above. Then the following two statements hold.

(i-1) for any v-generated submodule L $K_o = \ker d_o$ there exists an v-generated strongly pure submodule N^{Jn} of $P_n = P(I_n)$, v– generated strongly pure sub L^{Jo} of $P_o = P(I_o)$ and subsets $I'_0, I'_1, \ldots \ldots \ldots \ldots I'_n$ of $I_0, I_1, \ldots \ldots . I_n$ respectively, of cardinality $\leq v$ such that $d_j\left(P(I'_j)\right) \subseteq P(I'_{j-1})$ for j=1, 2 .....n. $N \subseteq N^{Jn} \subseteq K_n = K_n = \ker d_n, L \subseteq L^{Jo} \subseteq K_0 = \ker d_0$ for each $n \geq 1$, the sequence (* *) is strongly pure-exact where $P'_n = P(I'_j)$ and d'_n is the restriction of d_n to P'_j. In case n=0 we have $N^{J0} L^{J0}$.

(i-2) Assume that N, L, N^{Jn}, L^{Jo} and $I'_0, I'_1 \ldots \ldots \ldots \ldots I'_n$ are such that the statement (i-1) holds, and let N' and L' be v-generated submodules of K_n and K_o containing N and L, respectively. Then there exist an v-generated strongly pure submodule N^{Jn} of $P_n = P(I_n)$, v-generated strongly pure sub-module L'^{Jn} of $P_0 = P(I_0)$ and subset I_0'', I_1'', ... I_n'' of I_0, I_1, ... I_n respectively of cardinality $\leq v$ such that $d_j(P(I_j'')\ for\ j = 1, 2 \ldots \ldots n$

$N' \subseteq N'^{Jn} \subseteq K_n,\ L' \subseteq L'^{Jn} \subseteq K_n, J^{Jn} \subseteq N'^{Jn}\ L^{Jn} \subseteq L'^{Jn}$ the diagram

$$\begin{array}{ccccccccccccc}
O & \longrightarrow & N^{Jn} & \longrightarrow & P(I'_n) & \xrightarrow{d'_n} & P(F'_{n-1}) & \longrightarrow & \ldots \ldots \ldots . & \xrightarrow{d'_2} & P(I'_n) & \longrightarrow & L^{Jn} & \longrightarrow & O \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
O & \longrightarrow & N'^{Jn} & \longrightarrow & P(I''_n) & \xrightarrow{d''_n} & P(I''_{n-1}) & \longrightarrow & \ldots \ldots \ldots . & \xrightarrow{d''_2} & P(I''_n) & \xrightarrow{d''_1} & L'^{Jn} & \longrightarrow & O
\end{array}$$

is commutative and has strongly pure-exact rows, where the vertical arrows are natural embeddings induced by the inclusion.

$I_0' \subseteq I_0''$, $I_1'' \subseteq I_1'' \ldots\ldots\ldots I_n'' \subseteq I_n''$ and d_j'' is the restriction of d_j to $P(I_J'')$ for j=1,2 .....n. I_n case n=0 we have $N'^{J_0} = L'^{J_o}$

Assume that n=0 since the submodule N and L K_o are v-generated then applying Lemma 2.5 to the v-generated submodule N+L *of* K_o we get an v-generated strongly pure submodule $(N+L)^J$ *of* P (I_o) and K_o.

It follows that there is a subset I_o' *of* I_o Cardinality $\leq v$ such that $(N+L)^J$ is a strongly pure-submodule of $P(I_o') \subset P(I_o)$. It we set $N^{I_o} = L^{J_o} = (N+L)^J$ we get (i-1).

Proof of i-2: For any v-generated submodule Y of K_{n-1} and any submodule y of I_n of cardinality $\leq v$ contained y such that.

(ii-1) $d_n\big(p(y^1)\big) = Y^1$

(ii-2) The restriction $d_n': P(y') \longrightarrow Y^1$ of d_n to $P(\ y^1\ )$ is a strongly pure epimorphism and

(iii-3) the submodule ker d_n' of $P(Y')$ is v-generated.

Let Y be an *v-generated* submodule of K_{n-1} and y a subset of I_n of cardinality $\leq v$ and the module Y' as the direct sum $y^1 \bigoplus_{j=1}^{\omega} y_j$ of subsets (+) $y \subseteq y_1 \subseteq Y_2 \subseteq y_3 \ldots\ldots\ldots \subseteq y_j \subseteq y_{j+1} \subseteq \ldots\ldots..$ of I_n of cardinality $\leq v$ and the module Y' as the direct sum $y^1 \bigoplus_{j=1}^{\omega} y_j$ of v- generated strounly pure submodules.

(+ +) $y \subseteq y^{(1)} \subseteq y^{(2)} \subseteq \cdots \ldots\ldots \subseteq y^{(j)} \subseteq y^{(j+1)} \subseteq \ldots\ldots$ of K_{n-1} such that the image of the restriction $d^{(1)} : P\big(y(i)\big) \longrightarrow K_{n-1}$ of d_n of $p\big(y^{(j)}\big)$ such that $f = d^{(j)} f^1$.

We construct the sequence (+) and (++) strongly pure submodule $K=K_{n-1}$ of the strongly pure projective module $P=P(I_{n-1})$ and $L=Y$ we get an v-generated strongly pure submodule Y^J of K_{n-1} containing Y. We set $Y^{(J)}=Y^J$. By Lemma 2.6 the module $Y^{(1)}$ has a direct summand form.

$Y^{(1)}=\bigoplus_{\lambda\in\pi} Y_\lambda^{(1)}$ where π_1 is a set of cardinal its $\leq v$ and $Y_\lambda^{(1)}$is a countably generated strongly pure projective pure submodule of K_{n-1} for each $\lambda\in\pi_1$.

Since the epimorphism $d_n: P(I_n)\longrightarrow K_{n-1}$ is strongly pure and $Y_\lambda^{(1)}$is strongly pure projective then for each $\lambda\in\pi_1$the embedding $\phi_\lambda: Y_\lambda^{(1)}\longrightarrow Y^{(1)}$ has a factroisation $\phi=d_n f_\lambda$, where $f_\lambda\in Hom_R\left(Y_\lambda^{(1)}, P(I_n)\right)$ Since $f_\lambda\left(Y_\lambda^{(1)}\right)$ is a countably generated submodule of $P(I_n), |\pi|\leq v$ and $v-v_0$, then there exists a subset $y^{(1)}$ of I_n of cardinality $\leq v$ containing y such the $\bigoplus_{\lambda\in\pi_1} f_\lambda\left(Y_\lambda^{(1)}\right)\subseteq p\left(y^{(1)}\right)$. It follows that the image of the restriction $d^{(1)}: p\left(y^{(1)}\right)\longrightarrow K_{n-1}$of $d_n$to $p(y^{(11)})$ contains $Y^{(1)}\supset Y$, more over, for any finitely generated R-module Z and R-homomorphism $f: Z\longrightarrow Y^{(1)}$ there exists an R-homomorphism $f': Z\longrightarrow P\left(y^{(1)}\right)$ such that $f=d^{(1)}f'$. Indeed. I_{mf} is a finitely generated submodule of $Y^{(1)}$ and therefore there exists $\lambda\in\Omega$ such that $I_m f\subseteq Y_\lambda^{(1)}$. If we set $f^1=f_\lambda f$, we get the required equality $f=d^{(1)}f^1$. Hence we conclude $Y^{(1)}\subseteq I_m d^{(1)}$.

Since $|y^{(1)}|\leq v$, the submodule $I_m d^{(1)}$ $K_{n-1}$is v-generated and according to lemma 2.1 there exists an v-generated strongly pure submodule $\left(I_m d^{(1)}\right)^J$of K_{n-1}containing $I_m d^{(1)}$. We set $y^{(2)}=\left(I_m d^{(1)}\right)^J$. If $j\geq 1$ and $Y^{(J)}, y^{(j)}$are constructed, we construct $Y^{(j+1)}$ and $y^{(j+1)}$by applying the above construction of $Y^{(1)}, y^{(1)}$and $Y^{(2)}$ to $Y^{(1)}$ and the set $y^{(1)}$. The details are left to the reader.

Now we prove the inductive step. Assume that $n \geq 1$ and that statements (j – 1) and (j – 2) hold for N is an v-generated submodule of K_n and L is an v-generated submodule of K_o. We set $L_o = L$. By Lemma 2.5, $N_0^{J_n}$of $P(I_n)$ such that $N \subseteq N_o^{J_n} \subseteq K_n$.

Let $J'_{n,o}$be a subset of $I_n$of cardinality $\leq v$ such that $N_o^{J_n} \subseteq P(J'_{n,o}) \subseteq P(I_n)$.Then the submodule $T_o = d_n\left(P(J'_{n,o})\right)$of $K_{n-1} = \ker d_{n-1} \subseteq P(I_{n-1})$is v-generated. By applying the induction hypothesis to $T_o \subseteq K_{n-1}$and $L_o = L \subseteq K_o$ one get subsets $J_{n-1,0} \subseteq I_{n-1} \ldots . . J_{o,o} \subseteq I_o$of cardinality $\leq v$, and v −generated strongly pure sub module $T_o^{J_{n-1}} \subseteq K_{n-1} of\ P(J_{n-1,o})$ containing $T_o$and v −generated strongly pure submodule $L_o^{J_n} \subseteq K_o of\ P(J_{o,o})$ containing L_o such that the sequence

$$O \longrightarrow T_o^{J_{n-1}} \longrightarrow P(J_{n-1,o}) \xrightarrow{d_{n-1,o}} P(J_{n-2,o}) \longrightarrow \ldots \ldots \longrightarrow P(J_{1,o}) \xrightarrow{d_{n,o}} L_o^{J_o} \longrightarrow O$$

is strongly pure exact, where $d_{n,o}$ is the restrictuion of d_j to $P(J_{n,o})$ for $J=1,2,3.......n$-1.

By our claim applied to Y=$T_o^{J_{n-1}}$and $Y = J'_{n,o}$ there exist a subset $J_{n,o}$of I_n of cardinality $\leq v$ containing $J'_{n,o}$ and an v −generated stronly pure submodule $T_1 = \left(T_o^{J_{n-1}}\right)^1$ of K_{n-1}containing $T_o^{J_{n-1}}$such that $J'_{n,o} \subseteq J_{n,o}$the restriction of $d_n$to $P(J_{n,o})$ yields a strongly pure epimerphism.

$$d_{n,o}: P(J_{n,o}) \longrightarrow T_1$$

And the strongly pure submodule $\ker d_{n,o}$of $P(J_{n,o})$ is v −generated. It is clear that $N \subseteq N_o^{J_n} \subseteq \ker d_{n,o}$. By applying the induction hypothesis to $T_1 \subseteq K_{n-1}$and $L_1 = L_o^{J_o} \subseteq K_o$, one get submodule $J_{n-1,1} \subseteq I_{n-1} \ldots \ldots . J_{o,1} \subseteq I_o$ of cardinality $\leq v$ an v −generated strongly pure sub-module $T_1^{J_{n-1}} \subseteq K_{n-1}$of $P(J_{n-1,1})$ containing T_1, an v-generated strongly sub-module $L_1^{J_o} \subseteq K_o$of $P(J_{o,1})$containing L_1such that the sequence.

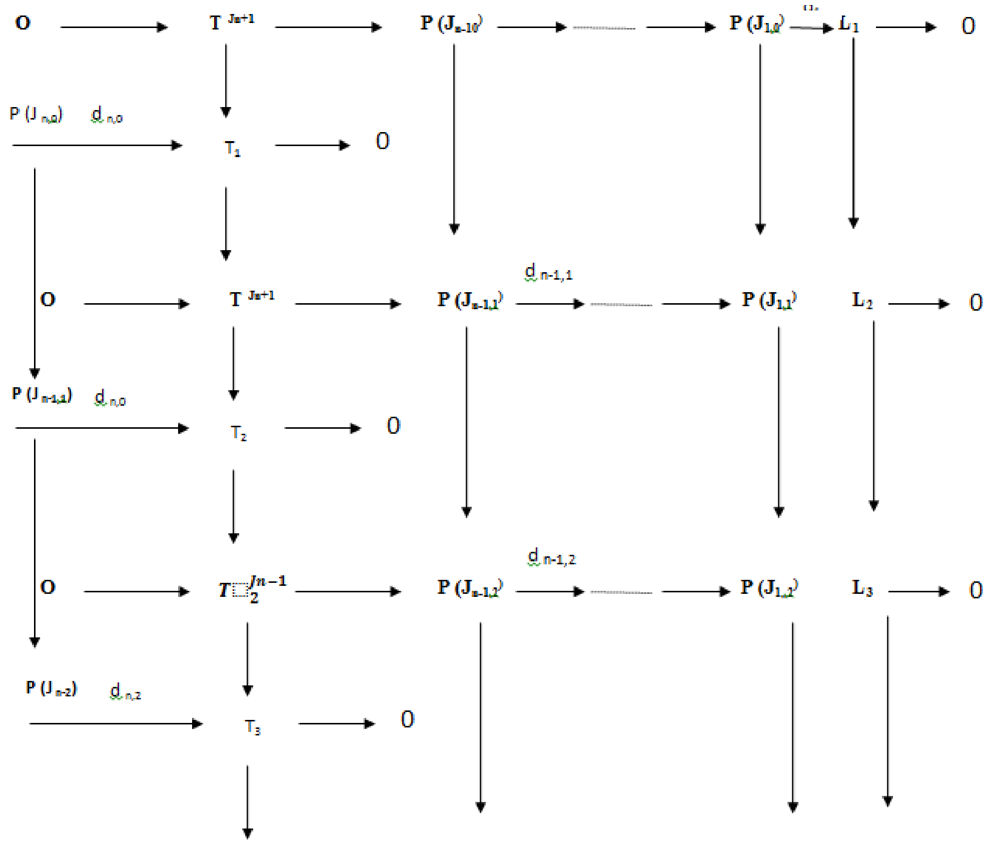
$$O \longrightarrow T_1^{J_{n-1}} \longrightarrow P(J_{n-1,o}) \xrightarrow{d_{n-1,1}} P(J_{n-2,1}) \longrightarrow \dots \dots \longrightarrow P(J_{1,1}) \xrightarrow{d_{1,1}} L_1^{J_o} \longrightarrow O$$

is strongly pure exact, where $d_j$to $P(J_{n,1})$ and $J_{j,o} \subseteq J_{j,o} \subseteq I_J$ for $j =$ $1,2 \dots n-1$ by our claim applied to $Y = T_1^{J_{n-1}}$and $Y = J_{n,o}$ there exist a subject $J_{n,1}$of $I_n$of K_{n-1} containing $T_1^{J_{n-1}}$ such that the restriction of d_n to $P(J_{n-1})$ yields a strongly pure epimorphism.

$d_{n,1}: P(J_{n-1}) \longrightarrow T_2$, the submodule ker $d_{n,1}$of $P(J_{n,1})$ is v-generated and $N \subseteq N_o^{J_n} \subseteq \ker d_{n,o} \subseteq \ker d_{n,1}$. Continuing this way, we construct two sequences.

- $T_o \subseteq T_o^{J_{n-1}} \subseteq T_1^{J_{n-1}} \subseteq \dots \dots \dots . \subseteq T_s \subseteq T_S^{J_{n-1}} \subseteq \dots \dots \dots.$
- $L = L_o \subseteq L_1 = L_o^{J_o} \subseteq I_\Omega = L_1^{J_o} \subseteq \dots \dots \dots \subseteq L_S = L_{S-1}^{J_o} \subseteq \dots \dots \dots.$

of v-generated sub modules of $K_{n-1} \subseteq P(I_{n-1}) and\ K_0 \subseteq$ $P(I_0)$ respectively and for each j $\subseteq$ {1,2.............n} chain $J_{j,0} \subseteq J\ J_{j,2} \subseteq J$............$\subseteq$ $J_{j,s}$ $\subseteq J_{s,s+I}$ $\subseteq$..................of subsects $T_S^{J_{n-1}} \subseteq P(J_{n-1,s})$ and Ls $\subseteq$ P $(J_{0,s-1})$ are strongly Pure embedding and the restriction of d_n to P $(J_{n,s})$ yields a strongly pure epimorphism $d_{n,s,}$: P $(J_{n,s}) \longrightarrow T_{s+1}$ It follows for each j $\in \{1,2 \dots . n\}$, there is a chain P $(J_{j,o}) \subseteq$ P $(J_{j1}) \subseteq$ P $(J_{j2}) \subseteq$.................. P $(J_{j+1}) \subseteq$ ....of submodules O $(J_{j,s})$ of (P (I,) and we get an infinite commutative diagram.



With stongly pure exact rows, where the vertical homorphism are the R-module embeddings constructed above. Let

$$0 \to N N^{J_n} \to P(I'_n) \xrightarrow{d'n'} P(I'_{n-1}) \xrightarrow{d'_{n-1'}} \ldots\ldots\ldots\ldots \xrightarrow{d'2} P(I'_1) \xrightarrow{d'1} L^{J_0} \to 0$$ be the

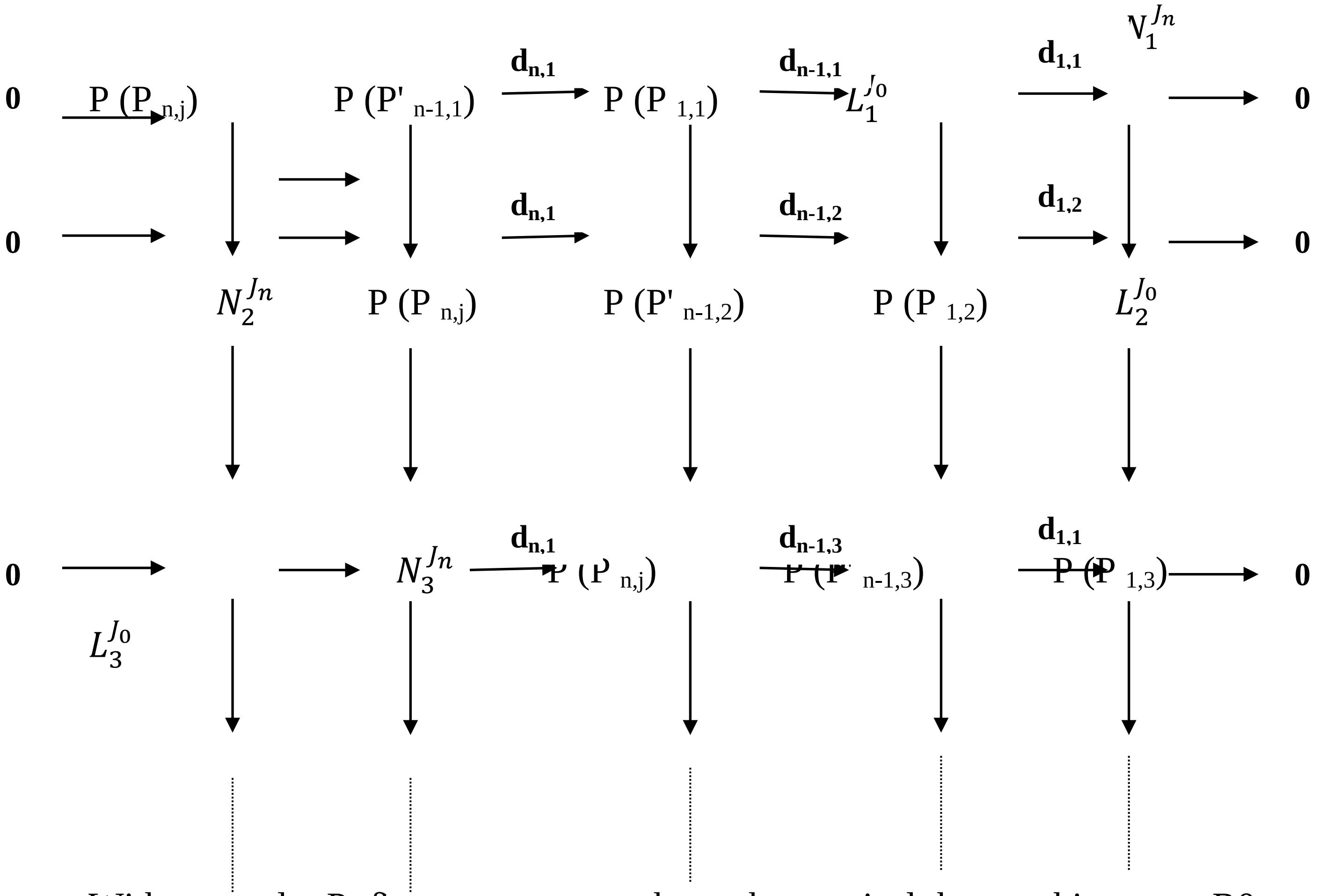
direct limit of the above system of strongly pure exact sequence where.

$N^{J_n} = \bigoplus_{s=1}^{\omega} Ker\, D_{n,s'}$ $N^{J_n} \bigoplus_{s=1}^{\omega} Ls$ and $I_J^I = \bigoplus_{s=0}^{\omega} j_{j,s}$ for j=1..............n, If follows the limit sequence is strongly pure- exact, wnsists of $^{v-}$ generated modules N^{Jn} = ker d_n' is a strongly pure submodule of P (I'_n) (and of Kn) containing N, the module.

$$I_m d'_n = \bigoplus_{s=1}^{\omega} T_{s} = \bigoplus_{s=1}^{\omega} T_S^{J^{n-1}} = kerd'_{n-1}$$

Is a strongly pure submodule of K_{n-1} and $L^{J_0} = \bigoplus_{s=1}^{\omega} Ls$ is a strongly pure submodule of P $(1'_0)$ as of K_0 By Lemma 2.6 the module N^{J_n}= kerd$'_n$ is v-generated.

(ii) Assume that n≥1 and $K_n \cong K_0$. Let N be an v- generated submodule of K_n and L on v- generated submodule of K_o. Fix an R - module isomorphism f:$k_n \to K_0$. Keeping the netation above and by applying (ii) we construct inductively an infinite commutative diagram.



With strongly Puδe exact rows, where the vertical homorphism are R0 module embedding induced by the inclusion I' $_{n,1} \subseteq I'_{j,3 \ldots\ldots\ldots\ldots for\ j=1,2 \ldots n}$ we set N_1= N+$^{f-1}$ (L) and h1 = f(N) + L if the modules $N_{j,}$ L_j and N_J^{J0} L_J^{J0} are defined set

$N_{j+1} = N_J^{J0} + f^{-1} (L_J^{J0})$ and $L_{n+1} = (N_J^{J0}) + L_J^{J0}$ It is clear that

$N_1 \subseteq N_J^{J0} \subseteq +_{n+1,} L, \subseteq (L_J^{J0}) \subseteq L_{j+1'}$ $f(N_1) =+ L_1$ and for j≥ 1we get $f(N_{n+1})= L_{j+1}$

Let $0 \rightarrow N^J \rightarrow P(I'_n) \xrightarrow{d_n} P(I'_{n-1}) \xrightarrow{\delta n-1} \ldots\ldots P(J'_n) \xrightarrow{\delta 1} L^J \rightarrow 0$ be the direct limit of the above system of strongly pure exact sequence where $N^J = \bigoplus_{s=1}^{\omega} N_s^{J0}\ L^J = \bigoplus_{s=1}^{\omega} L_s^j$ and $I_s^j = \bigoplus_{s=1}^{\omega} I_{j,s}^j$ for j= 1,2 ....n. It is easy to see that $f(N^J) = L^J$. Thus the modules N^J, L^J are isomorphic and the statements (ii) follows.

REFERNCES

[1] F.W. Anderson and K.R. Fuller, RIngs and Categories of Mudules, Springer-Verlag 1991, MR 941:16001, Zbl 0765,16001.

[2] M. Auslander, On the dimension of modules and algebras III. Global dimensions, Nagoya J.Math.. 9 (1995), 66-77, MR 17,579a, Zbl 0067.27103.

[3] S. Balcerzyx, On projective dimension of direct limit of modules. Bull.Acad. Polon. Sci., Ser. Sci. Math., 14 (1966), 241-244, MR 35#2946, Zbl 0158.28902.

[4] D.J. Benson, Flat modules over group rings of finite groups, Algebras and Represention Theory, 2 (1999), 287-294, MR 2000g:20013, Zbl 0938.20003.

[5] D.J. Benson and K.R. Goodearl, Periodic flat modules, and flat moduls for finite groups, Pacific K.Math, 196 (2000), 45-67, MR 2002c:20006.

[6] P.M. Cohn, On the free product of associative rings, Math. Z., 71 (1959), 380-398, MR 21#5648, Zbl 0087.26303.

[7] P. Griffith, on the decomposition of moduls and generalized left uniserial rings, Math Ann., 184 (1970), 300-308, MR 41#1790, Zbl 0184.06404.

[8] L. Grouson and C.U. Jensen, Lensen, L-dimension of rings and moduls, in 'Lecture Notes in Math., 67,' Springer Verlag, Berlin-Heidelberg-New York-Tokyo, 1980, 234-294.

[9] B. Huisgen-Zimmermann, Purity, algebraic compacteness, direct sum decompositions, and representation type, in 'infinite Length Moduls' (Eds. H. Krause and C.M. Ringel), Trends in Mathematics, Birkauser Verlag, Basel-Boston-Berlin, 2000. 331-367, MR 2002c:16008.

[10] C.U. Jensen, On homological dimensions of rings with countably generated ideals, Mat. Scand., 18 (1966), 97-105, MR 34#8711, Sbl 0145.26605.

[11] Les Foncteurs Derivers de lim et leurs Applications an Theorie des Moduls, Lecture Notes in Math., 258, Springer-Verlag, Berlin-Heidelberg-New York, 1972, MR 53#10874, Zbl 0238, 18007.

[12] C.U. Jensen and H. Lenzing, Model Theoretic Algebra With Particular Emphasis on Fields, Rings, Modules, Algebra, Algebra, Logic and Applications Vol. 2, Gordon & Breach Science Publishers, New York-London, 1989, MR 91m:03038,Zbl 0728.03026.

[13] I. Kaplansky. Projective moduls, Annals of Math., 68 (1958), 372-377, MR 20#6153, Zbl 0083.25802.

[14] R. Kieppinski, O Charkteryzacji I Strukturze Mudulow P-serwantic Injektywnych, Doctoral Thesis, Torun, 1967 (in Polish).

[15] On T-pure injective modules, Bull. Polon, Acad.Sci., Ser.Math., 15 (1967), 127-131, MR 36#3816, Zbl 0159.04602.

[16] R. Kielpinski and D.Simson, On pure homological dimension. Bill. Polon.Acad.Sci., Sr. Math., 23 (1975), 1-6 MR # 10872. Zbl 0303.16016.

[17] D. Lazard, Autour de la platitude, Bull. Soc. Math. France. 97 (1969), 81-128, MR 40 # 7310, Zbl 0174.33301.

[18] B. Mithcheli, Rings with several objects, Advances in Math., 8 (1972). 1-161, MR 45 # 3524, Zbl 0232.18009.

[19] B. Osofsky, Upper bounds on homological dimensions, Nagoya J. Math., 32 (1968), 315-322, MR 38 # 1128, Zbl 0162.05102.

[20] Projective dimensions of nice directed unions, J Pure Appl. Algebra. 13 (1978), 179-219, MR 81d:162.05102.

[21] N. Poperscu, Abelian Catagories With Applications to Rings and Moduls, Academic Press, 1973, MR 49 # 5130, Zbl 0271. 18006.

[22] D. Simson, flat and projective module, Bull. Polon. Acad. Sci., Ser. Math, 20 (1972), 109-114, MR 47 # 3450, Zbl 0232.16021.

[23] On Projective resolotions of flat modules, Colloq. Math., 29 (1974), 209-218, MR 50 # 13147, Zbl 0279. 16013.

[24] A remark on projective dimensions of flat modules, Math, Annalen, 209 (1974), 191-183, MR 49 # 9031. Zbl 0285.16024.

[25] On colinits of ijjectives in Grothendieck categories, Arkiv for Mat., 12 (1974), 161-165, MR 51 # 652, Zbl 0297.18003.

[26] On pure global dimension of locally finitely presented Grothendieck categories, Fund. Math., 96 (1977), 91-116, MR 58 # 845, Zbl 0361, 18010.

[27] Pure semisimple categories and rings of finite representation type. J. Algebra, 48, (1977), 290-296, MR 57 # 380, Zbl 0409.16030; Corrigendum, 67 (1980), 257-256, MR 82d:16028, Zbl 0449 0449. 16027.

[28] A class of potential counter-examples to the pure semisimplicity conjecture. Proc. Cont. 'Model Theory and Modules', Essen, 1994; Algebra, Logic and Applications, Vol. 9, Gordon & Breach Science

Publishers, Amsterdam, 1977, 345-373, MR 2000d:16019, Zbl 0936.16010.

[29] B. Stenstrom, Pure subdoduls, Arkiv for Mat., 7 (1967), 15-171, MR-36 # 6473, Zbl 0153.34302.

[30] Purity in functor categories, J. Algebra, 8 (1968, 352-361, MR 37 # 5271, Zbl 0157.04204.

[31] R.B. Warfield, Purity and algebraic compactness for modules, Pacific J. Math., 28 (1969), 699-719. MR # 4212, Zbl 0172-04801.