ERROR ESTIMATION OF SOLIDIFICATION PROCESS FOR INSIDE CYLINDRICAL VESSELS

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ABSTRACT

The present work considered solidifications of liquid inside cylindrical for five different value of latent heat parameter (α). The main aim of this work is to develop the approximate analytical formulas for solving problems of solidifications of liquids at their melting temperature(T_m) kept inside cylindrical vessels whose walls are maintained at constant wall temperature (T_W) such that (T_W<T_m) and to predict the solid-liquid interface position at any time. In this work the fifth order runga-kutta method has been used for solving the system of equation for the same problem obtained by the heat balance integral method incorporating special sub division. The result obtained by the above two methods have been compaired and an estimate of the error analysis has been carried out.

keywords:-solidification in cylindrical region, moving boundary problem, runga-kutta method, approximate method.

I. INTRODUCTION

There is a considerable concept in the group of problems known as Stefan's problem. There is comparatively little information of an analytical or approximate technique available and so the development of numerical method to deal with the Stefan's problems is highly desirable. Transient heat conduction problem involving melting and solidification generally referred to as "phase-change"or"moving-boundary" problems are important in many engineering applications such as in the making of ice, freezing of food, the solidification of metal in casting , and the, cooling of the large masses of the igneous rocks.The solution of this problem inherently difficult because the interface between the solid and the liquid phases is moving the latent heat absorbed or released at the interface.

Heat transfer problem involving the phase changes can be of two types.

(1.) When whole liquid is at its melting temperature that is no convection phenomenon in liquid region

(2.) The liquid is at higher temperature than its melting point and the convection phenomenon is included.

One of the most powerful solution techniques for the phase change problem is the heat balance integral method. In this method the heat conduction equation is approximated by an overall energy balance at regions of interest in both phases. With an assumed temperature profile an in conjunction with the energy balance condition at the phase front ,a system of differential equation are obtained and approximate solution for the interface location and other important physical parameter are than generated.

T.R.Goodman and Boston Mass[1] in 1958 presented n approximate mathematical method for solving heat- transfer problems utilizing the heat- balance integral and applied five problems involving a change of phase.

A refinement of the heat balance integral method as described by the G.E.Bell[2] in 1978 has been successfully applied to the problem of the solidification of the cylindrical pipe. G.E.Bell[3] in 1979 predicted the temperature distribution and the rate of removal of heat by a coolant for the process of a solidification of a liquid about a cold isothermal pipe. The author G.E.Bell[4] in 1979 has demonstrated how the incorporation of special subdivision overcomes the sensitivity previously observed in the heat balanced integral method. W.W.Yuen[5] in 1980 applied the heat-balanced integral method to melting problems with initial sub cooling

N.K Samaria[6] in 1987 studied solidification of liquid inside a spherical vessel using heat balance integral method incorporating special subdivisions.

R.S Gupta and Dhirendra Kumar[7] in 1981 extended the variable time step method. The same author R.S Gupta and Dhirendra Kumar[8] also soled using the MTVS method an unconventional moving boundary problem, this solution of a gas bubble in a liquid.

R.S Gupta and ambreesh Kumar[9]in 1984 presented the variable time step method for solving moving boundary problems by transforming the variable space domain. According to Sheelam mishra[10] in (2013), the aim is to develop the approximate analytical formulae for solving problems of solidification of liquids at their melting point and to predict the solid-liquid interface position at anytime by using fourth order Runga-Kutta method for the same problems obtained by the heat-balance integral method.

II. ANALYTICAL SOLUTIONS BY APPROXIMATE METHOD.

We considered a liquid being kept inside a cylindrical vessel whose wall are maintained at a constant temperature (T_W) such that (T_W<<T_m). Initially the liquid is at temperature T_1 , such that T_1 > $T_{m.}$ It is further assumed that the physical properties of the solidified material remain constant throughout the process and there is no change of volume on solidification. The process of heat conduction from liquid to wall starts occurring and after a lapse of time "t" the solid liquid interface reaches at a radius R(t) (which is a function of time t).

For liquid region

$$\frac{\partial^2 T_1(r,t)}{\partial r^2}+\frac{\Gamma}{r}\frac{\partial T_1(r,t)}{\partial r}=\frac{1}{a_1}\frac{\partial T_1(r,t)}{\partial t}$$

$0 < r < R(t), t > 0$ for inside solidification

For solidified region

$$\frac{\partial^2 T_s(r,t)}{\partial r^2} + \frac{\Gamma}{r}\frac{\partial T_s(r,t)}{\partial r} = \frac{1}{a_s}\frac{\partial T_s(r,t)}{\partial t}$$

$R(t) < r < a, t > 0$ for inside solidification.

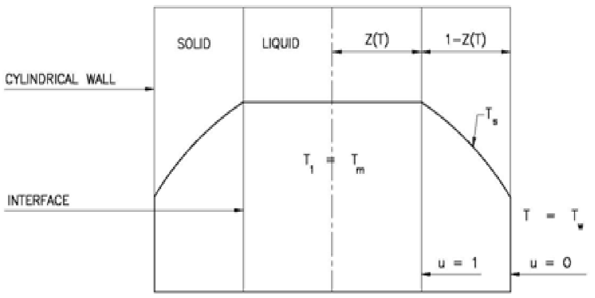

FIG. 2.1a :- SPECIFICATION OF CYLINDRICAL PROBL (INWARD SOLIDIFCATION)

The value of Γ is 1 for cylindrical vessels

The boundary condition for the general case for above solidification problem will be.
For inside solidification problem.

$$\frac{\partial T_l(r,t)}{\partial r} = 0, \quad \text{at } r = 0, t > 0$$

$$T_s(r,t) = T_l(r,t) = T_m \quad \text{at} \quad r = R(t), t > 0$$

$$T_s(r,t) = T_w \text{ at } \quad r = a, t > 0$$

At solid-liquid interface

$$K_S\frac{\partial T_s(r,t)}{\partial r} - K_l\frac{\partial T_l(r,t)}{\partial r} = \rho_s L_s\frac{dR(r,t)}{dt}$$

At $r = R(t), t > 0$ for inside cases of solidification.

So the initial conditions are.

At t=0, $T_l(r,t) = T_0 = T_m$ for all r, t=0 , R(0)=a

The above problem has been simplified by considering that the initial temperature T_m of the liquid is equal to constant, so the variation of temperature in the liquid region becomes zero. The equation of solidified region is applicable given by the simplified statement of the problem as:

$$\frac{\partial^2 T_s(r,t)}{\partial r^2} + \frac{\Gamma}{r}\frac{\partial T_s(r,t)}{\partial r} = \frac{1}{a_s}\frac{\partial T_s(r,t)}{\partial t}$$

So the boundary conditions are.
$R(t) < r < a$, t >0 for inside solidification.
$T_s(r,t) = T_w$ at r = a, t>0 for inside solidification problem.
$T_s(r,t) = T_m$ at r=R(t), t>0 for inside solidification problem.

$$K_S\frac{\partial T_s(r,t)}{\partial r} = \rho_s L_s\frac{dR(r,t)}{dt}$$

At $r = R(t), t > 0$ for inside cases of solidification .
The initial condition is given by.
$T_l(r,t) = T_m$ at t=0, at all r.
R(0) = a

The following non dimensional parameter has been adopted in order to generalize the problem so the equation becomes.

$$\frac{\partial^2 u(z,\tau)}{\partial z^2} + \frac{\Gamma}{z}\frac{\partial u(z,\tau)}{\partial z} = \frac{\partial u(z,\tau)}{\partial z} \qquad (1)$$

$Z(\tau) < z < 1, \tau > 0$ for inside solidification..

The boundary condition for inside cases of solidification process are given as.

$$u(z,\tau) = 0 \text{ at } z = 1, \tau > 0 \qquad (2)$$

$$u(z,\tau) = 1 \text{ at } z = (\tau), \tau > 0 \qquad (3)$$

$$\frac{\partial u(z,\tau)}{\partial z} = \alpha\frac{\partial z(\tau)}{\partial \tau} \quad \text{at } z = Z(\tau), \tau > 0 \qquad (4)$$

Where,

$$\boldsymbol{\alpha = \frac{1}{ste.} = \frac{L_s}{C_s(T_m - T_w)}}$$

α is non dimensional latent heat parameter with initial condition given as .

U(z,0)=1,atτ=0 for all z (5)

Z(0)=1

For approximate solution of the problem we assume that the time dependence of the temperature in the solidified region can be neglected.
So the temperature profile in the solidified region can be assumed to be stationary for the time being. This enables us to derive the analytical formulae for the temperature distribution in radial direction. From this temperature profile we get the value of temperature gradient $\partial u(z)/\partial z$ at $z=Z(\tau)$, and also we can get an expression for τ in terms of Z(τ).
Since,

$$\frac{\partial u(z)}{\partial \tau} \ll< \frac{\partial u(z)}{\partial z}$$

$$\therefore \frac{\partial u(z)}{\partial \tau} \cong 0$$

By applying the approximation the equation reduces to,

$$\frac{\partial^2 u(z,\tau)}{\partial z^2} + \frac{\Gamma}{z}\frac{\partial u(z,\tau)}{\partial z} = \frac{\partial u(z,\tau)}{\partial z} \qquad (6)$$

With boundary condition

u(z) = 0 at z = 1, >0 (7)

u(z) = 1 at z = Z(τ), >0 (8)

$$\frac{\partial u(z,\tau)}{\partial z} = \alpha\frac{\partial z(\tau)}{\partial \tau} \text{ at } z = Z(\tau), \tau > 0 \qquad (9)$$

And initial conditions,

u(z)=1at z=0 for all z (10)

z(0)=1

2.1 Cylindrical problem:

The analytical solution for cylindrical problem can be obtained from eqn. (6) by substituting Γ=1 with boundary conditions, (7) to (10) and is given as:

$$\frac{d^2u(z)}{dz^2}+\frac{1}{z}\frac{du(z)}{dz}=0 \qquad (11)$$

With boundary conditions,

$U(z)=0$ at $z=1$, $\Gamma>0$ (12)
$U(z)=1$ at $z=Z(\tau)$, $\tau>0$ (13)

$$\frac{du(z)}{dz}=\frac{dz(\tau)}{d\tau} \quad \text{at } z=Z(\tau),\ \tau>0 \qquad (14)$$

And initial conditions

$U(z)=1$ at $\tau=0$ for all z (15)

$Z(0)=1$
The solution for predicting interface position at any time can be formulated as follows:

$$\frac{d^2u(z)}{dz^2}+\frac{1}{z}\frac{du(z)}{dz}=0 \quad \text{Or} \quad z\frac{d^2u(z)}{dz^2}+\frac{1}{z}\frac{du(z)}{dz}=0$$

$$\frac{d}{dz}\left[z\frac{du(z)}{dz}\right]=0$$

on integrating above equation,

$$z\frac{du(z)}{dz}=C_1 \quad \text{or} \quad \frac{du(z)}{dz}=\frac{C_1}{z}$$

Again after integrating

$$u(z)=\int\frac{C_1}{z}dz+C_2=u(z)=C_1\ln+C_1$$

using the boundary condition,

$u(z)=1$ at $z=Z(\tau)$

$u(z)=0$ at $z=1$

we get

$C_2=0 \qquad C_1=\frac{1}{\ln Z(\tau)}$

$$\therefore u(z)=\frac{\ln z}{\ln Z(\tau)} \qquad (16)$$

After integrating eqn. (16), we get

$$\frac{du(z)}{dz}\Big|z=Z(\tau)=\frac{1}{Z(\tau)\ln Z(\tau)}$$

Also from eqn. (14) at interface

$$\frac{du(z)}{dz}\Big|z=Z(\tau)=\alpha\frac{dZ(\tau)}{d\tau}$$

$$\therefore\frac{1}{Z(\tau)\ln Z(\tau)}=\alpha\frac{dZ(\tau)}{d\tau}$$

Or $d\tau=\alpha\, Z(\tau)\ln Z(\tau)\, dZ(\tau)$

$$\tau=\alpha\int_1^{Z(\tau)}Z(\tau)\ln Z(\tau)dZ(\tau)$$

We know that

$$\int f(x)g(x)\,dz=f(x)\int g(x)dx-\int f(x)dx\int g(x)\,d(x)$$

Now, $\int\ln Z(\tau)\,dZ(\tau)=Z(\tau)\ln Z(\tau)-Z(\tau)$

$$\therefore\int Z(\tau)\ln Z(\tau)dZ(\tau)$$

$$=Z(\tau)[Z(\tau)\ln Z(\tau)-Z(\tau)]-\int 1[Z(\tau)\ln Z(\tau)-Z(\tau)]dZ(\tau)$$

$$=Z(\tau)^2\ln Z(\tau)-Z(\tau)^2-\int Z(\tau)\ln Z(\tau)dZ(\tau)+\int Z(\tau)dZ(\tau)$$

or $2\int Z(\tau)\ln Z(\tau)dZ(\tau)=Z(\tau)^2\ln Z(\tau)-Z(\tau)^2+Z(\tau)^2/2$

or $Z(\tau)\ln Z(\tau)dZ(\tau)=\frac{Z(\tau)^2\ln Z(\tau)}{2}-\frac{Z(\tau)^2}{4}$

or $\tau=\alpha\int_1^{Z(\tau)}Z(\tau)\ln Z(\tau)dZ(\tau)=\alpha\left[\frac{Z(\tau)^2\ln Z(\tau)}{2}-\frac{Z(\tau)^2}{4}\right]-\left\{-\frac{1}{4}\right\}$

$$=\alpha\frac{1}{4}[2Z(\tau)^2\ln Z(\tau)-Z(\tau)^2+1]$$

$$\because\tau=\alpha\int_1^{z(\tau)}Z(\tau)dZ(\tau)$$

$$\therefore\tau=\frac{\alpha}{4[2Z(\tau)^2\ln Z(\tau)-Z(\tau)^2+1]} \text{ at } Z=Z(\tau) \qquad (17)$$

Approximate time for complete solidification (when $Z(\tau)$=0) is given by (for inside solidification only). $\tau_s=\alpha/4=1/4$ ste.

The equation for heat flow rate through the wall of cylindrical vessel can be formulated as:

$Q=-K_sA(dT_s/dr)$

at r=a, z=r/a=1

$$Q=\frac{-2\pi K_s a(T_m-T_w)}{a}\frac{du(z)}{dz};$$

$$\left[\therefore\frac{du(z)}{dz}\Big|=\frac{du(z)}{dz}\right] \qquad (18)$$

$$\therefore Q=\frac{-2\pi K_s L_s}{C_s\alpha\ln Z(\tau)}$$

Eqn.(17) and (18) are applicable for case of inside solidification processes.

III. HEAT BALANCE INTEGRAL TRANSFORMATION TECHNIQUE:

In this technique the non dimensional temperature u from u=0 to 1 is sub divided into an equal, interval such that at i_{th} interval:

$u_i = i/n$, i= 0,1,2…………..n

A penetration variable is associated with each isotherm created by the sub division. The position of each isotherm u_i denoted by penetration variable z_i (τ)such that Z_0 (τ)=1 constant for all τ and Z_n (τ)=z(τ) is the position of solid liquid interface.

The equation (1) is reproduced as.

$$\frac{\partial^2 u}{\partial z^2}+\frac{\Gamma}{z}\frac{\partial u}{\partial z}=\frac{\partial u}{\partial \tau} \qquad (19)$$

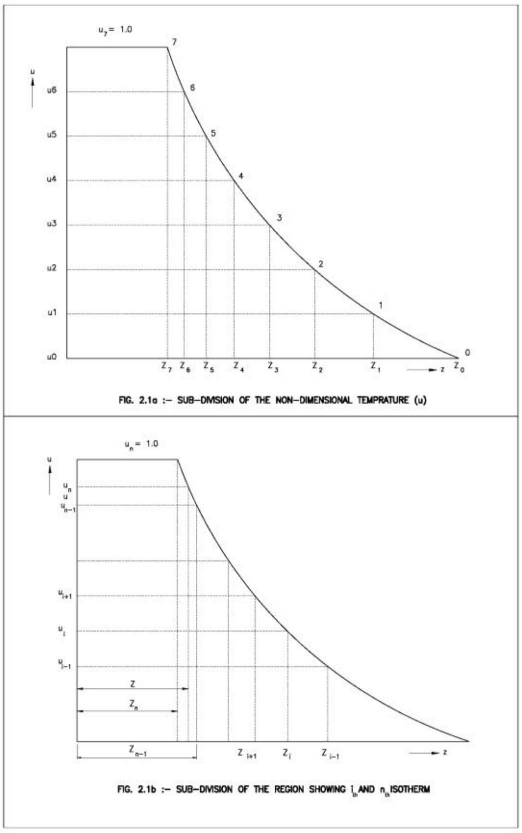

Figure 2. Specification of cylindrical vessels

$$\therefore u_{i+1}-u=\frac{1}{n}\,\frac{Z_{i+1}-z}{Z_{i+1}-Z_i}$$

$$u=u_{i+1}-\frac{1}{n}\,\frac{Z_{i+1}-z}{Z_{i+1}-z_i}$$

$$=\frac{i+1}{n}-\frac{1}{n}\,\frac{Z_{i+1}-z}{Z_{i+1}-z_i}$$

or

$$u=\frac{i}{n}+\frac{z-Z_i}{n(Z_{i+1}-Z_i)}$$

or

$$uz=\frac{iz}{n}+\frac{z(z-Z_i)}{n(Z_{i+1}-Z_i)}$$

$$\therefore \int_{z_i}^{z_{i+1}}(uz)\,dz=\int_{z_i}^{z_{i+1}}\left[\frac{iz}{n}+\frac{z^2-Z_i z}{n(Z_{i+1}-Z_i)}\right]dz$$

$$=\frac{i}{2n}\left[Z_{i+1}^2-Z_i^2\right]+\frac{1}{6n}\left[Z_{i+1}^2-Z_i^2-Z_{i+1}Z_i\right]$$

$$\therefore \int_{z_i}^{z_{i+1}}\frac{\partial}{\partial\tau}(uz)dz=\frac{i}{2n}\left[2Z_{i+1}Z_{i+1}-2Z_iZ_i\right]+\frac{1}{6n}\left[4Z_{i+1}Z_{i+1}-2Z_iZ_i-Z_{i+1}Z_i-Z_{i+1}Z_i\right]$$

$$\therefore \int_{z_i}^{z_{i+1}}\frac{\partial}{\partial\tau}(uz)dz=\int_{z_i}^{z_{i+1}}\frac{\partial}{\partial\tau}(uz)dz+\frac{i}{n}Z_iZ_i-\frac{i+1}{n}Z_{I+1}Z_{i+1} \qquad (20)$$

$$=\frac{i}{n}\left[Z_{i+1}Z_{i+1}-Z_iZ_i\right]+\frac{1}{6n}\left[4Z_{i+1}Z_{i+1}-2Z_iZ_i-Z_{i+1}Z_i-Z_{i+1}Z_i\right]$$

$$+\frac{i}{n}Z_iZ_i-\frac{i+1}{n}Z_{i+1}Z_{i+1}$$

$$\therefore \int_{z_i}^{z_{i+1}}\frac{\partial}{\partial\tau}(uz)dz=-\frac{1}{6n}\left[2Z_{i+1}Z_{i+1}+2Z_iZ_i+Z_{i+1}Z_i+Z_{i+1}Z_i\right] \qquad (21)$$

At the interface,

z=Z_n=Z(τ)

u_n=1 $\quad$ Z_0=1

$$\frac{\partial u}{\partial z}\Big|_{z=Z_n}=\alpha\frac{\partial Z_n}{\partial\tau}=\alpha Z_n$$

$$\int_{z_i}^{z_{i+1}}\frac{\partial}{\partial\tau}(uz)dz=\int_{z_i}^{z_{i+1}}\frac{\partial}{\partial\tau}(uz)dz+Z_{n-1}u_{n-1}Z_{n-1}-Z_nu_nZ_n \quad \text{from eqn. (20)}$$

From fig (2)

$$\frac{u_n-u_{n-1}}{u_n-u}=\frac{Z_n-Z_{n-1}}{Z_n-z}$$

or

$$u_n-u=\frac{1}{n}\,\frac{Z_n-z}{Z_n-Z_{n-1}}$$

or

$$u=u_n-\frac{1}{n}\,\frac{Z_n-z}{Z_n-Z_{n-1}}$$

or

$$uz=z-\frac{Z_nz-z^2}{n(Z_n-Z_{n-1})}$$

or

$$\int_{Z_{n-1}}^{Z_n}(uz)\,dz=\int_{Z_{n-1}}^{Z_n}\left[z-\frac{Z_nz-z^2}{n(Z_n-Z_{n-1})}\right]dz$$

$$\frac{1}{2}\left[Z_n^2-Z_{n-1}^2\right]-\frac{1}{n(Z_n-Z_{n-1})}\left[\frac{Z_n^2-Z_{n-1}^2}{2}-\frac{Z_n^3-Z_{n-1}^3}{3}\right]$$

$$\frac{\partial}{\partial\tau}\int_{Z_{n-1}}^{Z_n}(uz)\,dz = \frac{1}{2}[2Z_n Z_n - 2Z_{n-1}Z_{n-1}] - \frac{1}{6n}[2Z_n Z_n + Z_{n-1}Z_n + Z_n Z_{n-1} - 4Z_{n-1}Z_{n-1}]$$

$$[Z_n Z_n - Z_{n-1}Z_{n-1}] - \frac{1}{6n}[2Z_n Z_n + Z_{n-1}Z_n + Z_n Z_{n-1} - 4Z_{n-1}Z_{n-1}]$$

$$\therefore \frac{\partial}{\partial\tau}\int_{Z_{n-1}}^{Z_n}(uz)\,dz = \frac{\partial}{\partial\tau}\int_{Z_{n-1}}^{Z_n}(uz)\,dz + Z_{n-1}Z_{n-1}u_{n-1} - Z_n Z_n u_n$$

$$= [Z_n Z_n - Z_{n-1}Z_{n-1}] - \frac{1}{6n}[2Z_n Z_n + Z_{n-1}Z_n + Z_n Z_{n-1} - 4Z_{n-1}Z_{n-1}]$$

$$+\frac{n-1}{n}Z_{n-1}Z_{n-1}Z_n Z_n$$

$$\int_{Z_{n-1}}^{Z_n}\frac{\partial}{\partial\tau}(uz)\,dz = -\frac{1}{6n}[2Z_n Z_n + 2Z_{n-1}Z_{n-1} + Z_n Z_{n-1} + Z_{n-1}Z_n] \qquad (22)$$

3.1. Cylindrical problems:

From equation (19) we get the equation of temperature distribution for cylindrical problem by substituting Γ=1

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{z}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \tau z}$$

Multiply through out by z

$$z\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z} = z\frac{\partial u}{\partial \tau z}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\right] = z\frac{\partial u}{\partial \tau} = \frac{\partial(uz)}{\partial\tau}$$

Integrating over each sub region Z_i to Z_{i+1}, we get,

$$\int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial z}\left[z\frac{\partial u}{\partial z}\right]dz = \int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial \tau z}[uz]\,dz$$

$$\therefore \text{L.H.S} = \int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial \tau z}[uz]\,dz$$

$$= \left[Z_{i+1}\frac{\partial u}{\partial Z_{i+1}} - Z_i\frac{\partial u}{\partial Z_i}\right]$$

$$= Z_{i+1}\frac{1}{n(Z_{i+2}-Z_{i+1})} - Z_i\frac{1}{n(Z_{i+1}-Z_i)}$$

$$\text{L.H.S} = Z_{i+1}\frac{1}{n(Z_{i+2}-Z_{i+1})} - Z_i\frac{1}{n(Z_{i+1}-Z_i)}$$

$$\int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial\tau}(uz)\,dz = -\frac{1}{6n}[2Z_{i+1}Z_{i+1} + 2Z_i Z_i + Z_i Z_{i+1} + Z_{i+1}Z_i]$$

∵L.H.S=R.H.S

$$\therefore Z_{i+1}\frac{1}{n(Z_{i+2}-Z_{i+1})} - Z_i\frac{1}{n(Z_{i+1}-Z_i)} = -\frac{1}{6n}[2Z_{i+1}Z_{i+1} + 2Z_i Z_i + Z_i Z_{i+1} + Z_{i+1}Z_i]$$

From equation (23)

$$\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\right] = \frac{\partial(uz)}{\partial\tau}$$

Integrating above equation

$$\int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial z}\left[z\frac{\partial u}{\partial z}\right]dz = \int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial \tau z}[uz]\,dz$$

$$\therefore \text{L.H.S} = \int_{Z_i}^{Z_{i+1}}\frac{\partial}{\partial \tau z}[uz]\,dz$$

$$= \left[Z\frac{\partial u}{\partial z}\right]_{Z_{n-1}}^{Z_n} = Z_n\frac{\partial u}{\partial z}\Big|_{z=Z_n} - Z_{n-1}\frac{\partial u_{n-1}}{\partial Z_{n-1}}$$

$$\because \frac{\partial u}{\partial z}\Big|_{z=Z_n} = \alpha Z_n$$

$$\therefore \text{L.H.S} = Z_n\alpha Z_n - \frac{Z_{n-1}}{n(Z_n - Z_{n-1})}$$

$$\text{R.H.S} = \int_{Z_{n-1}}^{Z_n}\frac{\partial}{\partial\tau}(uz)\,dz = -\frac{1}{6n}[2Z_n Z_n + 2Z_{n-1}Z_{n-1} + Z_n Z_{n-1} + Z_{n-1}Z_n]$$

Equating L.H.S=R.H.S

$$= Z_n[(2+6n\alpha)Z_n + Z_{n-1}] + Z_{n-1}(2Z_{n-1} + Z_n)$$

$$= \frac{Z_{n-1}}{n(Z_n - Z_{n-1})}$$

(23)

For i = n-1

Thus following set of equations is obtained.

$$Z_i[2Z_1 + 1] = \frac{6}{Z_1 - 1} - \frac{6Z_1}{Z_2 - Z_1} \qquad \text{for } i = 0 \qquad (24)$$

$$Z_{i+1}[2Z_{i+1} + Z_i] + Z_i[2Z_i + Z_{i+1}] = \frac{6Z_i}{Z_{i+1} - Z_i} - \frac{6Z_{i+1}}{Z_{i+2} - Z_{1+1}} \qquad (25)$$

For i=1,2,3……….n-2

$$Z_n[(2+6n\alpha)Z_n + Z_{n-1}] + Z_{n-1}(2Z_{n-1} + Z_n) = \frac{6Z_{n-1}}{Z_n - Z_{n-1}} \qquad (26)$$

Heat flow rate through the wall of cylindrical vessel can be derived as:

At r = a , Z_0=1

$$Q = -K_s A\frac{\partial T}{r}$$

$$Q = -K_s A\frac{\partial T_s}{r}\Big|_{r=a}$$

$$= -2\pi K_s(T_m - T_w)\frac{\partial u}{\partial z}\Big|_{z=Z_0} \qquad \because (Z_0 = 1)$$

$$+2\pi K_s(T_m - T_w)/n(1 - Z_1)$$

$$\square = \frac{2\pi K_s L_s}{nC_s\alpha(1 - Z_1)} \qquad (27)$$

Equation (24) to (27) are applicable for inside solidification processes

3.2 Statement of the problem:

We have choosen cylindrical vessel of radius a= 28 cm = 0.28m and have consider its unit length (1m). Water has been choosen as liquid at 0^0C and the problem of ice formation has been tackled for five different values of wall temperature i:e, T_w= -8^0C, -10^0C, -20^0C, -30^0C and -40^0C.

The properties of ice at 0^0C has been taken as:

K_s=2.2x10^{-3}	KW/Mk
ρ_s=913	Kg/m^3
C_s=1.93	KJ/Kg K
a_s=1.26x10^{-6}	m^2/sec
L_s= 334.88	KJ/Kg

From these properties of ice the four different values of latent heat parameter has been calculated for five different values of wall temperature'T_w' by the formula.

$$\alpha = \frac{L_s}{[C_s(T_m - T_w)]}$$

we have sub divided u=0 to 1 into seven equal intervals.
For the purpose of solving the problems of cylindrical vessel the starting values of Z_7 has been taken as 0.95 for inside solidification.

3.3 Cylindrical problems:

The value of another penetration variable (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6) have been calculated by the approximate analytical method. (from the eqn. (16).

$$\therefore u(z) = \frac{\ln z}{\ln Z(\tau)} = \frac{\ln z}{\ln Z_7} \quad \text{for inside problems.}$$

Starting time has been calculated corresponding to different value of α by the eqn. (17)

$$\tau_0 = \frac{\alpha}{4}\left[2Z^2(\tau)\ln Z(\tau) - Z^2(\tau) + 1\right] \quad \text{for inside case}$$

IV. USE OF RUNGA-KUTTA FOURTH ORDER METHOD:

Numerical solution for the system of equation obtained from the heat balanced integral method for cylindrical, inside solidification process is obtained by using fourth order runga kutta method.

We sub divided u=0 to 1 into 7 equal intervals such that the at i_{th} interval u_i=i/7 and i=0,1,2……..7

4.1. Cylindrical problems:

From equations (25) and (26), we have

$$Z_{i+1}[2Z_{i+1} + Z_i] + Z_i[2Z_i + Z_{i+1}] = \frac{6Z_i}{Z_{i+1} - Z_i} - \frac{6Z_{i+1}}{Z_{i+2} - Z_{1+1}}$$

For i=1,2,3…….6

$$Z_n[(2 + 6n\alpha)Z_n + Z_{n-1}] + Z_{n-1}(2Z_{n-1} + Z_n) = \frac{6Z_{n-1}}{Z_n - Z_{n-1}} \quad \text{for } n = 7$$

For the sake of computational convenience we have simplified above equations by
following substitutions.

Δt is denoted by T

Z_{i+1} is denoted by Z(J)

Z_n is denoted by Z(J) where i=J-1

$$Z_{i+1} = \frac{dZ_{i+1}}{d\tau} \quad \text{is denoted by ZD(J)}$$

After these substitution, applying runga-kutta method (for j=2to8),

We get,

$$\text{Let } A(J) = \frac{6Z(J-1)}{Z(J) - Z(J-1)}$$

B(J)=2Z(J)+Z(J-1)=P.Z(J).W(J)

Where P=6nα=6x7α=42α

W(J)=0 for J=2 to 7

C(J)=2Z(J-1)+Z(J)

∴ZD(J)=[A(J)-A(J+1)-ZD(J-1).C(J)]/B(J)

Where A(J+1)=0 for J=8

Q1(J)=ZD(J).T

ZA(J)=Z(J)+Q1(J)/2

$$A1(J) = \frac{6ZA(J-1)}{ZA(J) - ZA(J-1)}$$

B1(J)=2ZA(J)+ZA(J-1)+P.ZA(J).W(J)

C1(J)=2ZA(J-1)+ZA(J)

∴ZAD(J)=[A1(J)-A1(J+1)-ZAD(J-1).C1(J)]/B1(J)

Where A1(J+1)=0 for J=8

Q2(J)=ZAD(J).T

ZB(J)=Z(J)+Q2(J)/2

$$A2(J) = \frac{6ZB(J-1)}{ZB(J) - ZB(J-1)}$$

B2(J)=2ZB(J)+ZB(J-1)=P.ZB(J).W(J)

C2(J)=2ZB(J-1)+ZB(J)

∴ZBD(J)=[A2(J)-A2(J+1)-ZBD(J-1).C2(J)]/B2(J)
Where A2(J+1)=0 for J=8

Q3(J)=ZBD(J).T

ZC(J)=Z(J)+Q3(J)/2

$$A3(J) = \frac{6ZC(J-1)}{ZC(J) - ZC(J-1)}$$

B3(J)=2ZC(J)+ZC(J-1)=P.ZC(J).W(J)

C3(J)=2ZC(J-1)+ZC(J)

∴ZCD(J)=[A3(J)-A3(J+1)-ZCD(J-1).C3(J)]/B3(J)

Where A3(J+1)=0 for J=8

Q4(J)=ZCD(J).T

$$\therefore Z(J)|_{T_0+T} = Z(J)|_{T_0} + \frac{Q1(J) + 2Q2(J) + 2Q3(J) + Q4(J)}{6}$$

Which is applicable for inside solidification process.

V. USE OF RUNGA KUTTA FIFTH ORDER METHOD

Numerical solution for the system of equation obtained from the heat balanced integral method for cylindrical , inside solidification process is obtained by using fifth order runga-kutta method.We sub divided u=0 to 1 into 7 equal intervals such that the at i_{th} interval u_i=i/7 and i=0,1,2……..7

5.1 Cylindrical problems:

From equations (25) and (26), we have

$$Z_{i+1}[2Z_{i+1} + Z_i] + Z_i[2Z_i + Z_{i+1}] = \frac{6Z_i}{Z_{i+1} - Z_i} - \frac{6Z_{i+1}}{Z_{i+2} - Z_{1+1}}$$

For i=1,2,3…….6

$$Z_n[(2 + 6n\alpha)Z_n + Z_{n-1}] + Z_{n-1}(2Z_{n-1} + Z_n) = \frac{6Z_{n-1}}{Z_n - Z_{n-1}} \quad \text{for } n = 7$$

For the sake of computational convenience we have simplified above equations by following substitutions.

Δt is denoted by T

Z_{i+1} is denoted by Z(J)

Z_n is denoted by Z(J) where i=J-1

$$Z_{i+1} = \frac{dZ_{i+1}}{d\tau} \quad \text{is denoted by ZD(J)}$$

After these substitution, applying runga-kutta method (for j=2to8),

We get,

$$\text{Let } A(J) = \frac{6Z(J-1)}{Z(J) - Z(J-1)}$$

B(J)=2Z(J)+Z(J-1)=P.Z(J).W(J)

Where P=6nα=6x7α=42α

W(J)=0 for J=2 to 7

C(J)=2Z(J-1)+Z(J)

∴ZD(J)=[A(J)-A(J+1)-ZD(J-1).C(J)]/B(J)

Where A(J+1)=0 for J=8

Q1(J)=ZD(J).T

ZA(J)=Z(J)+Q1(J)/2

$$A1(J) = \frac{6ZA(J-1)}{ZA(J) - ZA(J-1)}$$

B1(J)=2ZA(J)+ZA(J-1)+P.ZA(J).W(J)

C1(J)=2ZA(J-1)+ZA(J)

∴ZAD(J)=[A1(J)-A1(J+1)-ZAD(J-1).C1(J)]/B1(J)

Where A1(J+1)=0 for J=8

Q2(J)=ZAD(J).T

ZB(J)=Z(J)+Q2(J)/2

$$A2(J) = \frac{6ZB(J-1)}{ZB(J) - ZB(J-1)}$$

B2(J)=2ZB(J)+ZB(J-1)=P.ZB(J).W(J)

C2(J)=2ZB(J-1)+ZB(J)

∴ZBD(J)=[A2(J)-A2(J+1)-ZBD(J-1).C2(J)]/B2(J)
Where A2(J+1)=0 for J=8

Q3(J)=ZBD(J).T

ZC(J)=Z(J)+Q3(J)/2

$$A3(J) = \frac{6ZC(J-1)}{ZC(J) - ZC(J-1)}$$

B3(J)=2ZC(J)+ZC(J-1)=P.ZC(J).W(J)

C3(J)=2ZC(J-1)+ZC(J)

∴ZCD(J)=[A3(J)-A3(J+1)-ZCD(J-1).C3(J)]/B3(J)

Where A3(J+1)=0 for J=8

Q4(J)=ZCD(J).T

ZD(J)=Z(J)+Q4(J)/2

$$A4(J) = \frac{6ZD(J-1)}{ZD(J) - ZD(J-1)}$$

B4(J)=2ZD(J)+ZD(J-1)=P.ZD(J).W(J)

C4(J)=2ZD(J-1)+ZCD(J)

∴ZDD(J)=[A4(J)-A4(J+1)-ZDD(J-1).C4(J)]/B4(J)

Where A4(J+1)=0 for J=8

Q5(J)=ZDD(J).T

ZE(J)=Z(J)+Q5(J)/2

$$A5(J) = \frac{6ZE(J-1)}{ZE(J) - ZE(J-1)}$$

B5(J)=2ZE(J)+ZE(J-1)=P.ZE(J).W(J)

C4(J)=2ZE(J-1)+ZED(J)

∴ZED(J)=[A5(J)-A5(J+1)-ZED(J-1).C5(J)]/B5(J)

Where A5(J+1)=0 for J=8

Q6(J)=ZED(J).T

ZF(J)=Z(J)+Q6(J)/2

$$A6(J) = \frac{6ZF(J-1)}{ZF(J) - ZF(J-1)}$$

B6(J)=2ZF(J)+ZF(J-1)=P.ZF(J).W(J)

C6(J)=2ZF(J-1)+ZFD(J)

∴ZFD(J)=[A6(J)-A6(J+1)-ZFD(J-1).C6(J)]/B6(J)

Where A6(J+1)=0 for J=8

$$\therefore Z(J)|_{T_0+T} = Z(J)|_{T_0} + \frac{7Q1(J) + 32Q3(J) + 12Q4(J) + 32Q5(j) + 7Q6(j)}{90}$$

Which is applicable for inside solidification process.

VI. CONCLUSION

The main of this work is to pridict the time required for the complition of a solidification process and also to pridict the temperature distribution in the solidified region of a liquid kept inside cylindrical whose wall are maintained and constant sub melting temperature. The heat balance integral technique incorporating special subdivision has been used and the obtained system of equation has been solved by fifth order rungaa kutta method. Although a number of assumption have been made in the problem specification and change of flux has been approximated by the discontinues change in the adjacent profile gradiants. The acceptable estimate of both the temperture and the flux have been obtained by using small subdivision (n=7) and incremental time (initial t=2x10^{-12}). An approximate analytical approach has been made to pridict solid liquid interface position at any time. Results obtained by both numerical and analytical method, have been compaired and an estimate of the error has been established, and by applying Runga-Kutta fifth method to be the more correct.The result show that the percentage of error while using approximate analytical method is less for higher values of latent heat parameter and higher for lower values. The percentage error ε=0% at α=9.3 for inside solidification process of cylindrical vessel respectively. The percentage error becomes infinite when α is reduced below 4.0 for cylindrical inside vessel inside case. Thus , approximate analytical solution can be adopted for higher values of α with a maximum error encountered which has been shown in the table 1.

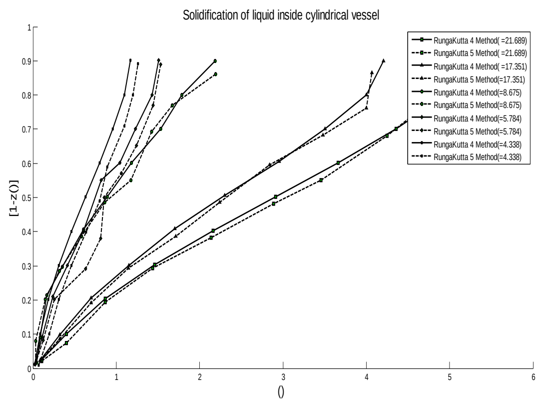

Figure 3. Comparison of variation of solidified region with time for different values of latent heat parameter (α) for cylindrical vessel.

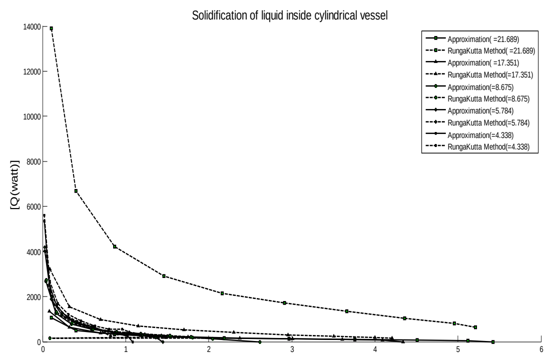

Figure 4. Comparison of rate of heat out through the wall of cylindrical vessel for different values of latent heat parameter (α).

The result show that the percentage of error while using approximate analytical method is less for higher values of latent heat parameter and higher for lower values. The percentage error ε=0% at α=9.3 and ε=0% at α=10.0 for inside solidification process of cylindrical vessel respectively.

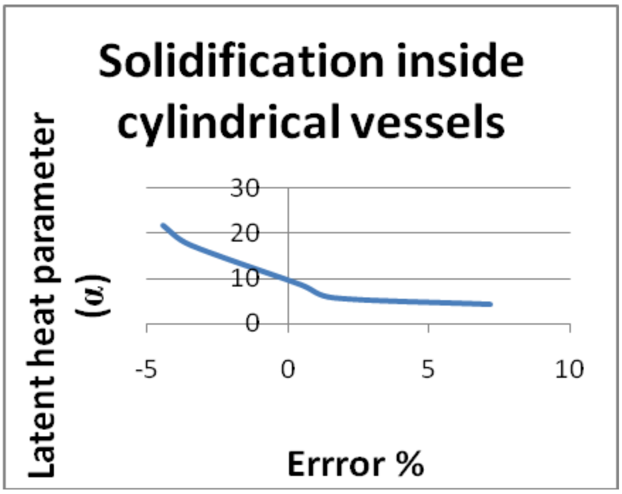

Figure 5. Variation of percentage errors in complete solidification time for different values of latent heat parameter (α) for cylindrical vessels.

Table 1. Complete solidification times for different values of latent heat parameters (α) by approximate methods, Runga-Kutta fourth and Runga-Kutta fifth order method for inside cylindrical solidification processes.

A	τ_{sr4}	τ_{sa}	τ_{rk5}	% error(εR_4)	%error (εr_5)
4.338	1.17	1.08	1.26	7.69	7.14
5.784	1.51	1.47	1.53	2.65	1.62
8.675	2.18	2.17	2.19	0.46	0.45
17.351	4.20	4.34	4.06	-3.33	-3.44
21.689	5.20	5.42	4.98	-4.23	-4.41

REFERENCES

[1]. T.R.Goodman, Boston.Mass, the heat balance integral and its application to problem involving a change of phase,pp.335-342(1958).
[2]. G.E.Bell, A refinement of the heat balance integral method applied to a melting problem, Int.J.HMT, Vol.21,pp.1357-1362(1978).
[3]. Graham E.Bell, solidification of a liquid about,acylindricalpipe,Int.J.HMT,Vol.22,pp1681-1685(1979)
[4]. G.E.Bell , solidification of a liquide about a cylindrical pipe, Int.J.HMT, Vol.22,pp.1681-1685(1979).
[5]. W.W.Yuen,shorter communications, applications of the heat balance integral to melting problems with initial subcooling, Int.J.HMT, Vol.23,pp.1157-1160(1980).
[6]. R.S.Gupta and Dheerendra Kumar, variable times step methods for one dimensional stefan problem with mixed boundary condition, Int.J.HMT,Vol.24,pp.251-259,(1981).
[7]. Radhey S.Gupta and Dhirendra kumar treatment of the solidification time inside and outside cylinders by variable time step methods , Int.J.HMT,Vol.26,pp.313-315,(1983).
[8]. Radhey S.Gupta and Ambreesh kumar, variable time step methods with coordinate transformation, computer method in applied mathematics and engineering, 44, pp 91-103(1984).
[9]. N.K Samaria, solidification of liquid inside a spherical vessel, proceeding 8 nat.HMT. Conf.pp.343-347(1987).
[10].Sheelam Mishra (2013),develop the approximate analytical formulae for solidification, I.J.A.I.S.T, Vol 26, No 26, pp.12-17(2013).

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