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A STUDY ON $(E,q)(E,q)$ PRODUCT SUMMABILITY

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Abstract

This paper introduces the concept of $(E,q)(E,q)$ product summability of Fourier series and its Conjugate Fourier series. Under a general condition, we have determine two new theorems on the same operator as a double summability .

Keywords: (E,q) summability, $(E,q)(E,q)$ summability.

Introduction

The product summability $(E,q)(X)$, $(X)(E,q)$ or $|E,q|$ of Fourier series & its allied series, have been studied by a number of researchers like, Prasad, Kanhaiya [6], Nigam, H.K. [5], Chandra, P. [2], Chandra, P. and Dikshit, G.D. [1], Tiwari, Sandeep kumar, and Bariwal, Chandrashekhar [8], Dhakal, Binod Prasad [3]. Also a lot of work has been carried out by Mohanty, R. and Mohapatra, S. (1968), Kwee, B. (1972), Sachan, M.P. (1983), Bhagwat, Purnima (1987), Lal, S., Singh, H.P., Tiwari, Sandeep kumar, and Bariwal, Rathore, H.L. and Shrivastava, U.K. (2012), Nigam, H.K. and Sharma, K. (2012,2013), Sinha, Santosh kumar & Shrivastava, U.K. (2014), Mishra, V.N. and Sonavare, Vaishali (2015) and many more, under analogous conditions. In the same line, so many results established on double factorable summability of double Fourier series, the methods of $(C,1,1)$, $(H,1,1)$ and (N,p_m,q_n). Till now, no result found on double Euler summability of Fourier series & its allied series as a general case. Under a general condition, hear we have established two new theorems on $(E,q)(E,q)$ product summability of Fourier series and its Conjugate series.

Definition and Notation

Let $f(t)$ be a Fourier series integrable in the Lebesgue sense over $(-\pi,\pi)$ and periodic with period 2π, then let

$$f(t)\sim\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=1}^{\infty}A_n(t) \tag{2.1}$$

and

$$\sum_{n=1}^{\infty}(a_n\sin nt-b_n\cos nt)=-\sum_{n=1}^{\infty}B_n(t) \tag{2.2}$$

is called the Conjugate series of Fourier series.

Let $\sum_{n=0}^{\infty}a_n$ be an infinite series whose n^{th} partial sum s_n is given by $s_n-\sum_{v=0}^{n}a_v$.

If

$$E_n^{\,q}=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k\to s \text{ as } n\to\infty$$

then the infinite series $\sum_{n=0}^{\infty}a_n$ is said to be (E,q) summable to a definite number s, (Hardy [4]).

The product of (E,q) summability by itself defines the $(E,q)(E,q)$ double summability. Thus the $(E,q)(E,q)$ transform of $t_n^{E^qE^q}$ of $\{s_n\}$ is given by

$$t_n^{E^qE^q}=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}s_v\right\}$$

If $t_n^{E^qE^q}\to s$, as $n\to\infty$ then the series $\sum_{n=0}^{\infty}a_n$ is summable to s by $(E,q)(E,q)$ summability method.

We use the following notation throughout the paper.

$$\phi(t)=f(x+t)+f(x-t)-2f(x)$$
$$\psi(t)=f(x+t)-f(x-t)$$

$$\left(E^q E^q\right)_n(t)=\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin t/2}$$

$$\left(\tilde{E}^q \tilde{E}^q\right)_n(t)=\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\cos(v+1/2)t}{\sin t/2}$$

$\tau=\left[\frac{1}{t}\right]$ is integral part of $\frac{1}{t}$

Known Theorem

In 2013, Nigam, H.K.[5] has proved the following theorems on $(C,2)(E,1)$ summability of Fourier series and its Conjugate series.

Theorem 1: Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$C_n=\sum_{v}^{n}c_v\rightarrow\infty$, as $n\rightarrow\infty$

If

$$\Phi(t)=\int_0^t|\phi(u)|du=o\left[\frac{t}{\alpha(1/t)C_\tau}\right]\text{as } t\rightarrow+0$$

(3.1)

where, $\alpha(t)$ is positive, monotonic and non-increasing function of t

and $\log(n+1)=O[\{\alpha(n+1)\}C_{n+1}]$, as $n\rightarrow\infty$

(3.2)

then the Fourier series (2.1) is summable $(C,2)(E,1)$ to $f(x)$.

Theorem 2: Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$C_n=\sum_{v}^{n}c_v\rightarrow\infty$, as $n\rightarrow\infty$

If

$$\Psi(t)=\int_0^t|\psi(u)|du=o\left[\frac{t}{\alpha(1/t)C_\tau}\right]\text{, as } t\rightarrow+0$$

(3.3)

where $\alpha(t)$ is positive, monotonic and non-increasing function of t,

$$2^\tau\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)=O(n+1)(n+2)$$

(3.4)

And condition (3.2) holds then the Conjugate Fourier series (2.2) is summable to

$$\bar{f}(x)=-\frac{1}{2\pi}\int_0^{2\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt$$

at every pt where this integral exists.

Main Theorem

Theorem 1: Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$P_n=\sum_{v=0}^{n}p_v\rightarrow\infty$, as $n\rightarrow\infty$

If

$$\Phi(t)=\int_0^t|\phi(u)|du=o\left[\frac{t}{\log(1/t)}\right]\text{, as } t\rightarrow+0$$

(4.1)

then the Fourier series (2.1) is summable $(E,q)(E,q)$ to $f(x)$ at pt $t=x$.

Theorem 2: Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$P_n=\sum_{v=0}^{n}p_v\rightarrow\infty$, as $n\rightarrow\infty$

If

$$\Psi(t)=\int_0^t|\psi(u)|du=o\left[\frac{t}{\log(1/t)}\right]\text{as } t\rightarrow+0$$

(4.2)

then the Conjugate Fourier series (2.2) is summable $(E,q)(E,q)$ to

$$\tilde{f}(x)=-\frac{1}{2\pi}\int_0^{2\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt$$

at every pt, where this integral exists.

5. Lemmas

For the proof of the theorem, we require the following lemmas.

Lemma 1: If

$$\left(E^q E^q\right)_n(t)=\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin t/2}$$

then $\left|\left(E^q E^q\right)_n(t)\right| = O(n+1)$ for $0 \le t \le \pi/n+1$

Proof:

$$\left|\left(E^q E^q\right)_n(t)\right| \le \left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin t/2}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}(2v+1)$$

$$= \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{(2k+1)}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}$$

$$= \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(2k+1)$$

$$= \frac{1}{2\pi(1+q)^n}(2n+1)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}$$

$$= \frac{2n+1}{2\pi}$$

$$= O(n+1) \qquad (5.1)$$

Lemma2: If

$$\left(E^q E^q\right)_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin t/2}$$

then $\left|\left(E^q E^q\right)_n(t)\right| = O\left(\frac{1}{t}\right)$ for $\frac{\pi}{n+1} \le t \le \pi$

Proof:

$$\left|\left(E^q E^q\right)_n(t)\right| \le \left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin t/2}\right|$$

$$\le \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}$$

$$= \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}$$

$$= O\left(\frac{1}{t}\right) \qquad (5.2)$$

Lemma3: If

$$\left(\tilde{E}^q \tilde{E}^q\right)_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\cos(v+1/2)t}{\sin t/2}$$

then $\left|\left(\tilde{E}^q \tilde{E}^q\right)_n(t)\right| = O\left(\frac{1}{t}\right)$ for $0 \le t \le \frac{\pi}{n+1}$

Proof:

$$\left|\left(\tilde{E}^q \tilde{E}^q\right)_n(t)\right| \le \left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\cos(v+1/2)t}{\sin t/2}\right|$$

$$= \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}$$

$$= \frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}$$

$$= O\left(\frac{1}{t}\right) \qquad (5.3)$$

Lemma4: If

$$\left(\tilde{E}^q \tilde{E}^q\right)_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\cos(v+1/2)t}{\sin t/2}$$

then $\left|\left(\tilde{E}^q \tilde{E}^q\right)_n(t)\right| = O\left(\frac{1}{t}\right)$ for $\frac{\pi}{n+1} \le t \le \pi$

Proof:

$$\left|\left(\tilde{E}^q \tilde{E}^q\right)_n(t)\right| \le \left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\cos(v+1/2)t}{\sin t/2}\right|$$

$$\le \left|\frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\ \mathrm{Re}\,\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}e^{i(v+1/2)t}\right|$$

$$\le \left|\frac{1}{2t(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\ \mathrm{Re}\,\frac{1}{(1+q)^k}\left\{\sum_{v=0}^{k}\binom{k}{v}q^{k-v}e^{ivt}\right\}\right|\left|e^{it/2}\right|$$

$$\leq \left| \frac{1}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \operatorname{Re} \frac{1}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} e^{ivt} \right\} \right|$$

$$\leq \left| \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k} \operatorname{Re} \frac{1}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} e^{ivt} \right\} \right| + \left| \frac{1}{2t(1+q)^n} \sum_{k=\tau}^{n} \binom{n}{k} q^{n-k} \operatorname{Re} \frac{1}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} e^{ivt} \right\} \right|$$

$$= J_1 + J_2$$

$$|J_1| \leq \left| \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k} \operatorname{Re} \frac{1}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} e^{ivt} \right\} \right|$$

$$\leq \left| \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k} \frac{1}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \right\} \right| \left| e^{ivt} \right|$$

$$\leq \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k} \frac{1}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v}$$

$$= \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k}$$

$$= O\left(\frac{1}{t}\right)$$

Now,

$$|J_2| \leq \left| \frac{1}{2t(1+q)^n} \sum_{k=\tau}^{n} \binom{n}{k} q^{n-k} \operatorname{Re} \frac{1}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} e^{ivt} \right\} \right|$$

$$\leq \frac{1}{2t(1+q)^n} \sum_{k=\tau}^{n} \binom{n}{k} q^{n-k} \frac{1}{(1+q)^k} \max_{0 \leq m \leq k} \left| \sum_{v=0}^{k} \binom{k}{v} q^{k-v} e^{ivt} \right|$$

$$= O\left(\frac{1}{t}\right)$$

(5.4)

Proof

Proof of Theorem 1: Following Zygmund [9], the n^{th} partial sum $s_n(x)$ of the series (2.1) at $t = x$ is given by

$$s_n(x) = f(x) + \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin(n+1/2)t}{\sin(t/2)} dt.$$

Therefore, (E,q) transform of (E,q) is given by

$$t_n^{E^q E^q} - f(x) = \int_0^{\pi} \phi(t) \left(E^q E^q\right)_n(t) dt$$

$$= \int_0^{\pi/n+1} \phi(t) \left(E^q E^q\right)_n(t) dt + \int_{\pi/n+1}^{\pi} \phi(t) \left(E^q E^q\right)_n(t) dt$$

$$= I_{1.1} + I_{1.2}, \quad \text{(say)}$$

(6.1)

We have

$$|I_{1.1}| = \int_0^{\pi/n+1} |\phi(t)| \left| \left(E^q E^q\right)_n(t) \right| dt$$

$$= O(n+1) \left[\int_0^{\pi/n+1} |\phi(t)| dt \right] \qquad \text{by (5.1)}$$

$$= O(n+1) \left[o\left\{ \frac{1}{(n+1)\log(n+1)} \right\} \right]$$

$$= o\left\{ \frac{1}{\log(n+1)} \right\}$$

$$= o(1), \text{ as } n \to \infty$$

(6.2)

$$|I_{1.2}| = \int_{\pi/n+1}^{\delta} |\phi(t)| \left| \left(E^q E^q\right)_n(t) \right| dt$$

$$= O\left[\int_{\pi/n+1}^{\delta} |\phi(t)| \left(\frac{1}{t}\right) dt \right] \qquad \text{by (5.2)}$$

$$= O\left[\left\{ \frac{1}{t} \Phi(t) \right\}_{\pi/n+1}^{\delta} + \int_{\pi/n+1}^{\delta} \frac{1}{t^2} \Phi(t) dt \right]$$

$$= O\left[o\left\{ \frac{1}{\log(1/t)} \right\}_{\pi/n+1}^{\delta} + \int_{\pi/n+1}^{\delta} o\left\{ \frac{1}{t \log(1/t)} \right\} dt \right]$$

$$= o\left\{ \frac{1}{\log(n+1)} \right\} + o(1) \left[-\log\log(1/t) \right]_{\pi/n+1}^{\delta}$$

$$= o(1) + o(1), \text{ as } n \to \infty$$

$$= o(1), \text{ as } n \to \infty$$

(6.3)

From (6.2) and (6.3) we have

$$t_n^{E^q E^q} - f(x) = o(1) \text{ as } n \to \infty$$

This completes the proof of theorem 1.

Proof of Theorem2:

$$\tilde{t}_n^{E^q E^q} - \tilde{f}(x) = \int_0^{\pi} \psi(t) \left(\tilde{E}^q \tilde{E}^q\right)_n(t) dt$$

$$= \int_0^{\pi/n+1} \psi(t)\left(\tilde{E}^q \tilde{E}^q\right)_n(t)dt + \int_{\pi/n+1}^{\pi} \psi(t)\left(\tilde{E}^q \tilde{E}^q\right)_n(t)dt$$

$$= K_{1.1} + K_{1.2} \text{ (say)} \quad (6.4)$$

Let

$$\left|K_{1.1}\right| = \int_0^{\pi/n+1} |\psi(t)| \left|\left(\tilde{E}^q \tilde{E}^q\right)_n(t)\right| dt$$

$$= O\left[\int_0^{\pi/n+1} \frac{1}{t}|\psi(t)|dt\right] \quad \text{by (5.3)}$$

$$= O(n+1)\left[\int_0^{\pi/n+1} |\psi(t)|dt\right]$$

$$= O(n+1)\left[o\left\{\frac{1}{(n+1)\log(n+1)}\right\}\right]$$

$$= o\left\{\frac{1}{\log(n+1)}\right\}$$

$$= o(1), \text{ as } n \to \infty \quad (6.5)$$

$$\left|K_{1.2}\right| = \int_{\pi/n+1}^{\delta} |\psi(t)| \left|\left(\tilde{E}^q \tilde{E}^q\right)_n(t)\right| dt$$

$$= O\left[\int_{\pi/n+1}^{\delta} \frac{1}{t}|\psi(t)|dt\right] \quad \text{by (5.4)}$$

$$= O\left[\left\{\frac{1}{t}\Psi(t)\right\}_{\pi/n+1}^{\delta} + \int_{\pi/n+1}^{\delta} \frac{1}{t^2}\Psi(t)dt\right]$$

$$= O\left[o\left\{\frac{1}{\log(1/t)}\right\}_{\pi/n+1}^{\delta} + \int_{\pi/n+1}^{\delta} o\left\{\frac{1}{t\log(1/t)}\right\}dt\right]$$

$$= o\left\{\frac{1}{\log(n+1)}\right\} + o\{1\}\left[-\log\log(1/t)\right]_{\pi/n+1}^{\delta}$$

$$= o(1) + o(1) \text{ as } n \to \infty$$

$$= o(1) \text{ as } n \to \infty \quad (6.6)$$

Combining (6.5) and (6.6) we have

$$\tilde{t}_n^{E^q E^q} - \tilde{f}(x) = o(1) \text{ as } n \to \infty$$

This completes the proof of theorem 2.

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